TECHNICAL REPORT R-99

THEORY OF THE SECULAR VARIATIONS IN THE ORBIT OF A SATELLITE OF AN OBLATE PLANET

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SUMMARY

The theory of satellite orbits about an oblate planet is studied by means of a new set of canonical variables. The Hamiltonian function is separated into two parts, one of which is neglected. The neglected part is periodic with mean value equal to zero, and it vanishes when the inclination is zero. Thus the solution obtained by neglecting this part of the Hamiltonian is exact for equatorial orbits; for inclined orbits the secular motion of the node and perigee are obtained correctly to the second order in the oblateness parameter.

For satellite orbits the geometric equation of the trajectory is obtained in the classical form in terms of true and eccentric anomalies, with these being related to the physical angle (argument of latitude) by transformations involving elliptic functions. The kinematic equation obtained is a natural generalization of Kepler's equation. All the orbital elements are constants in this approximation; the perturbation equations for the elements are exhibited but not solved.

A numerical example is included based on the satellite 1958 $\beta 2$ (Vanguard 1). Secular motions are predicted accurately, and periodic motions within the limitations of the theory.

Relativistic effects are shown to be negligible as far as the geometry of the orbit is concerned, while the secular drift of a satellite-borne clock is shown to be on the fringe of detectability.

INTRODUCTION

The theory of satellite orbits about an oblate planet has been discussed by many authors in recent years, using a variety of methods. Brouwer (ref. 1) has obtained a solution by starting with the elliptical orbit and then computing the effects of oblateness by von Zeipel's modification of Delaunay's method. Garfinkel (ref. 2) uses the same method but starts with an intermediate orbit, obtained by means of an approximate potential function (for the planet's gravitational field) that leads to separability of the Hamilton-Jacobi equation in spherical polar coordinates. Vinti (ref. 3) obtains an implicit solution in closed form by solving the Hamilton-Jacobi equation in ellipsoidal coordinates. Kozai (ref. 4) begins with the elliptical orbit and then applies Lagrange's method of variation of parameters.

The purpose of the present paper is to obtain a new intermediate orbit having three important properties. First, it is the complete solution in the equatorial case. Second, in the case of inclined orbits, the secular motion of the node and perigee are treated correctly to the second order in the oblateness parameter. Third, the elements of the orbit are displayed explicitly in the form of rapidly converging series involving the oblateness parameter, which can easily be carried to any desired order of accuracy and are well adapted to numerical computation. Thus, the solution presented here is more tractable than Vinti's and is more accurate than the others in the equatorial case.

Finally, the perturbation equations relating the intermediate orbit to the complete problem are obtained but not solved, simply to put the present solution in its proper perspective.

SYMBOLS

a	semimajor axis of orbit, dimensionless
A	coefficient in orbit equation, appendix D
A_m	coefficient in Fourier series expansion of elliptic integral
B	angular momentum in relativistic equations

B_m coefficients in Fourier series for $\frac{d\theta}{dv}$
c speed of light
C coefficient used to make $\theta=0$ at perigee
C_m coefficients in Fourier series for cos v
D dissipative force
D dimensionless dissipative force, $\frac{\mathbf{D}R^2}{m\mu}$
E total energy in relativistic equations
E eccentric anomaly
E complete elliptical integral of second kind
E_i generalized force
f frequency parameter in elliptic functions
F generating function in Hamilton-Jacobi theory
F distance-dependent portion of generating function
F force
F_i generalized force
$g(w)$ characteristic cubic
G universal constant of gravitation
$h(u)$ characteristic cubic
H Hamiltonian function
H_0 secular portion of Hamiltonian function
H_1 periodic portion of Hamiltonian function
I angle of inclination of orbital plane to earth's equatorial plane
J_k dimensionless coefficients in earth's gravitational potential
J $\frac{3J_2}{2}$, Jeffreys' coefficient
k modulus of elliptic functions
k' complementary modulus
$K=K(k)$ complete elliptic integral of first kind
$K'=K(k')$ complete elliptic integral with complementary modulus
L Lagrangian function
m mass of satellite
M mass of planet
M mean anomaly
M_m coefficients in series expansion of mean anomaly (Kepler's equation)
n mean motion
$O(\ )$ order of magnitude
p_i, P_i generalized momenta
p_ρ radial velocity, dimensionless
$p_\varphi=P_\varphi$ angular momentum, dimensionless
$p_\Omega=P_\Omega$ axial component of angular momentum, dimensionless
P_ρ total energy
p semilatus rectum, dimensionless
q_i, Q_i generalized coordinates
q Jacobi's nome
$-Q_\rho=\tau_p$ time of perigee passage
$Q_\varphi=\varphi_p$ argument of perigee
$Q_\Omega=\Omega_p$ right ascension of ascending node at time of perigee passage
R planet's equatorial radius
r geocentric distance
S dimensionless disturbance function $\frac{\Phi}{\mu/R}$
S_0 secular portion of S
S_1 periodic portion of S
T kinetic energy, dimensionless
t time, proper time
u reciprocal geocentric distance, dimensionless
u_1, u_2, u_3 roots of characteristic equation
U negative Newtonian potential
v true anomaly
v_m coefficients in Fourier series for v
V velocity
w auxiliary variable used in Cardan's solution of the cubic
w_1, w_2, w_3 roots of characteristic equation
W velocity, dimensionless, $V\sqrt{R/\mu}$
X_i, Y_i generalized forces
α geocentric right ascension
$\alpha_j, \beta_j, \gamma_j, \delta_j$ definite integrals used in derivation of Kepler's equation, appendix E
β azimuth of velocity vector, clockwise from north
γ flight-path angle, upward from horizontal
δ geocentric declination
Δ discriminant of characteristic equation
ϵ eccentricity of orbit
ϵ_0 parameter related to eccentricity
ζ auxiliary angle used in Cardan's solution of the cubic
η total energy, dimensionless
η_1, η_2 limiting values of η for positive discriminant
η_T value of η for which orbit is tangent to surface of planet
θ auxiliary angle used as independent variable; equals right ascension for

	equatorial orbits, declination for polar orbits
$\Delta\theta$, $\Delta\varphi$	advance of perigee per revolution
θ_m	coefficients in Fourier series for v
ϑ, ϑm	auxiliary angle and coefficients in Fourier series for v
λ	σ/ξ^2, independent variable in series expansions
μ	$G(M+m)$, dynamical constant of gravitation
ν	μ/Rc^2, relativistic oblateness parameter, dimensionless
ρ	r/R, dimensionless geocentric distance
ξ	$\frac{1}{2}P_\varphi^2$, dimensionless
σ, σ_4	oblateness-inclination parameters, dimensionless
σ'	oblateness-inclination-relativistic parameter, dimensionless
τ	dimensionless time, $t\sqrt{\mu/R^3}$
τ_p	time of perigee passage
Φ	disturbance potential
φ	argument of latitude
φ_p	argument of perigee
φ_s, φ_m	secular and harmonic coefficients in Fourier series for φ
ψ	auxiliary angle in elliptic functions
ω	earth's spin velocity
Ω	right ascension of ascending node
$\Delta\Omega$	advance of ascending node per revolution
Ω_p	right ascension of ascending node at time of perigee passage
Ω_s, Ω_m	secular and harmonic coefficients in Fourier series for Ω
$\binom{m}{n}$	binomial coefficient
$[x, y]$	Lagrangian bracket
$^\circ$	degrees of arc
$''$	seconds of arc
$\lvert\;\rvert$	absolute value
$(\dot{})$	$d/d\tau$

SUBSCRIPTS

a	conditions at apogee
p	conditions at perigee
o	initial conditions
i	referred to inertial frame
e	referred to earth
$r, \alpha, \delta, \varphi, I$	components of vector in direction of increase of variable denoted by subscript

ORBITAL COORDINATES AND THE EQUATIONS OF MOTION

The orbital coordinates to be used here were introduced (in a different notation) by the present author in reference 5, and are shown in figure 1.

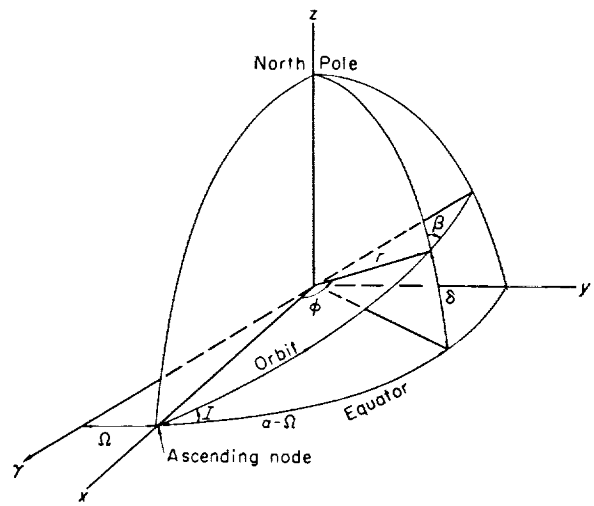

FIGURE 1.—Coordinate systems.

The equations of motion (ref. 5, eqs. (33), (34)) are

$$\frac{d^2r}{dt^2}-\frac{V_\varphi^2}{r}=\frac{F_r}{m}$$

$$\frac{d(rV_\varphi)}{dt}=\frac{rF_\varphi}{m}$$

$$\frac{d\Omega}{dt}=\frac{F_I\sin\varphi}{mV_\varphi\sin I}$$

$$\frac{dI}{dt}=\sin I\operatorname{ctn}\varphi\frac{d\Omega}{dt}$$

$$V_\varphi=r\left(\frac{d\varphi}{dt}+\frac{d\Omega}{dt}\cos I\right)$$

where F_r, F_φ, and F_I are the force components in the direction of increasing r, φ, and I, respectively. It is convenient to separate them as follows:

$$\frac{F_r}{m}=-\frac{\mu}{r^2}+\frac{\partial\Phi}{\partial r}+\frac{\mathbf{D}_r}{m}$$

$$\frac{F_\varphi}{m}=\frac{1}{r}\frac{\partial\Phi}{\partial\varphi}+\frac{\mathbf{D}_\varphi}{m}$$

$$\frac{F_I}{m}=\frac{1}{r\sin\varphi}\frac{\partial\Phi}{\partial I}+\frac{\mathbf{D}_I}{m}$$

where Φ is the disturbance potential (due, for example, to oblateness) and $\mathbf{D}_r$, $\mathbf{D}_\varphi$, $\mathbf{D}_I$ are forces not derivable from a potential (for example, air drag).

It is convenient to introduce dimensionless variables as follows:

$$\left.\begin{aligned} \rho&=\frac{r}{R} & D_\rho&=\frac{\mathbf{D}_r R^2}{m\mu}\\ \tau&=t\sqrt{\mu/R^3} & D_\varphi&=\frac{\mathbf{D}_\varphi R^2}{m\mu}\\ W_\varphi&=V_\varphi\sqrt{R/\mu} & & \\ S&=\frac{\Phi R}{\mu} & D_I&=\frac{\mathbf{D}_I R^2}{m\mu}\end{aligned}\right\} \tag{1}$$

where R is the equatorial radius of the planet, t is time measured in seconds, V_φ is the horizontal component of velocity, and $\mu=G(M+m)$, where G is the universal constant of gravitation, M is the mass of the planet, and m is the mass of the satellite.

Adopting the conventional values for the earth,

$$R=6378.388\text{ km}\qquad \mu=398632.9\text{ km}^3/\text{sec}^2$$

and assuming that m/M is negligible, gives 13.44710 minutes as the unit of τ and 7.905532 km/sec as the unit of W.

The equations of motion can now be written in the dimensionless form

$$\left.\begin{aligned} &\frac{d^2\rho}{d\tau^2}-\frac{W_\varphi^2}{\rho}=-\frac{1}{\rho^2}+\frac{\partial S}{\partial\rho}+D_\rho\\ &\frac{d(\rho W_\varphi)}{d\tau}=\frac{\partial S}{\partial\varphi}+\rho D_\varphi\\ &\frac{d\Omega}{d\tau}=\frac{1}{W_\varphi\sin I}\left(\frac{1}{\rho}\frac{\partial S}{\partial I}+D_I\sin\varphi\right)\\ &\frac{dI}{d\tau}=\sin I\text{ ctn }\varphi\,\frac{d\Omega}{d\tau}\\ &W_\varphi=\rho\left(\frac{d\varphi}{d\tau}+\frac{d\Omega}{d\tau}\cos I\right)\end{aligned}\right\} \tag{2}$$

EQUATIONS OF MOTION IN LAGRANGIAN FORM

Define the Lagrangian, L, by

$$L=T+\frac{1}{\rho}+S$$

where the kinetic energy, T, is given by

$$T=\frac{1}{2}\dot\rho^2+\frac{1}{2}W_\varphi^{\;2}$$

and the dot denotes differentiation with respect to τ. Then the equations of motion can be transformed into the four-dimensional Lagrangian form

$$\left.\begin{aligned} &\frac{d}{d\tau}\left(\frac{\partial L}{\partial\dot\rho}\right)-\frac{\partial L}{\partial\rho}=D_\rho\\ &\frac{d}{d\tau}\left(\frac{\partial L}{\partial\dot\varphi}\right)-\frac{\partial L}{\partial\varphi}=\rho D_\varphi\\ &\frac{d}{d\tau}\left(\frac{\partial L}{\partial\dot I}\right)-\frac{\partial L}{\partial I}=\rho D_I\sin\varphi\\ &\frac{d}{d\tau}\left(\frac{\partial L}{\partial\dot\Omega}\right)-\frac{\partial L}{\partial\Omega}=\rho(D_\varphi\cos I-D_I\sin I\cos\varphi)\end{aligned}\right\} \tag{3}$$

The derivation of the first three is straightforward; the fourth requires the relation

$$\frac{\partial S}{\partial\Omega}=\frac{\partial S}{\partial\varphi}\cos I-\frac{\partial S}{\partial I}\sin I\text{ ctn }\varphi \tag{4}$$

which is valid for any function of position

$$S(\rho,\varphi,\Omega,I)=S(\rho,\alpha,\delta)$$

This relation is proved in appendix A.

EQUATIONS OF MOTION IN CANONICAL FORM

Since the Lagrangian, L, does not contain $\dot I$, it follows that I is an ignorable coordinate. Following the usual procedure, consider the remaining coordinates, ρ, φ, Ω, and the conjugate momenta, defined by

$$\left.\begin{aligned} p_\rho&=\frac{\partial L}{\partial\dot\rho}=\dot\rho\\ p_\varphi&=\frac{\partial L}{\partial\dot\varphi}=\rho W_\varphi=\rho^2(\dot\varphi+\dot\Omega\cos I)\\ p_\Omega&=\frac{\partial L}{\partial\dot\Omega}=\rho W_\varphi\cos I=p_\varphi\cos I\end{aligned}\right\} \tag{5}$$

so that I is given by

$$I=\text{arc cos }(p_\Omega/p_\varphi) \tag{6}$$

Next introduce the Hamiltonian (the total energy):

$$H=2T-L=\frac{1}{2}p_\rho^{\,2}+\frac{1}{2}\left(\frac{p_\varphi}{\rho}\right)^2-\frac{1}{\rho}-S(\rho,\varphi,\Omega,I) \tag{7}$$

with I to be replaced by equation (6), so that H is expressed explicitly as a function of the coordinates ρ, φ, Ω and momenta p_ρ, p_φ, p_Ω.

It is again a straightforward process to transform the equations into the canonical form

$$\left.\begin{aligned}
\frac{d\rho}{d\tau}&=\frac{\partial H}{\partial p_\rho}\\
\frac{d\varphi}{d\tau}&=\frac{\partial H}{\partial p_\varphi}-\frac{\rho p_\Omega D_I \sin\varphi}{p_\varphi^2 \sin I}\\
\frac{d\Omega}{d\tau}&=\frac{\partial H}{\partial p_\Omega}+\frac{\rho D_I \sin\varphi}{p_\varphi \sin I}\\
\frac{dp_\rho}{d\tau}&=-\frac{\partial H}{\partial \rho}+D_\rho\\
\frac{dp_\varphi}{d\tau}&=-\frac{\partial H}{\partial \varphi}+\rho D_\varphi\\
\frac{dp_\Omega}{d\tau}&=-\frac{\partial H}{\partial \Omega}+\rho(D_\varphi \cos I-D_I \sin I\cos\varphi)
\end{aligned}\right\} \tag{8}$$

Thus, in the absence of dissipative forces, these equations are in canonical, Hamiltonian form. This means that the enormous body of literature on the subjects of contact transformations, variation of parameters, and the Hamilton-Jacobi theory is available for their solution.

It should be remarked that equations (8) are a completely general formulation, valid for any problem in celestial mechanics, since the only restrictions are that the dominant force be an inverse-square central force (this is the term $-1/\rho^2$ in the first of eqs. (2)) and that the disturbance function, S be a function of position only (see appendix A).

In the case of an oblate planet, the disturbance function, S, is usually written in the form (ref. 1, p. 396).

$$S=-\sum_{k=2}^{\infty} J_k P_k(\sin\delta)/\rho^{k+1} \tag{9}$$

where P_k is the Legendre polynomial of degree k, and (see appendix A)

$$\sin\delta=\sin I\sin\varphi$$

Following Brouwer, the case in which S is truncated at $k=2$ will be called the "main" problem. In principle, additional terms can be handled by perturbation methods.

It may be remarked that the definition of the Lagrangian is not unique. Because of the relation between $\dot{I}$ and $\dot{\Omega}$ (the fourth of eqs. (2)), the horizontal velocity, W_φ, could be expressed in terms of $\dot{I}$ instead of $\dot{\Omega}$. Then Ω would be the ignorable coordinate. The present procedure is preferred for two reasons. First, the ignorable coordinate, I, is given directly by equation (6) without an additional integration. In the alternative procedure the ignorable coordinate, Ω, could be obtained only by integrating the equation for its derivative. Secondly, the purpose here is to study motion about an oblate planet, with a disturbance function of the form given by equation (9). Now this form assumes axial symmetry. Thus, the disturbance force lies entirely in the meridian plane, and has no east-west component. Hence one integral of the equations is known immediately; namely, the component of angular momentum along the planet's axis is constant. In the present notation, this means that the conjugate momentum p_Ω is constant. This enters the analysis in a natural way and reduces the order of the system of equations immediately.

THE MAIN PROBLEM OF ARTIFICIAL SATELLITE THEORY

For the main problem the disturbance function, S, reduces to

$$S=\frac{J}{\rho^3}\left(\frac{1}{3}-\sin^2 I\sin^2\varphi\right) \tag{10}$$

where

$$J=\frac{3}{2}J_2$$

For the earth, $J=0.0016232$ (ref. 6), when R is taken to be the equatorial radius. Now S can be separated into two parts, only one of which is dependent on φ:

$$S=S_0+S_1$$

where

$$S_0=\frac{J}{3\rho^3}\left(1-\frac{3}{2}\sin^2 I\right)$$

$$S_1=\frac{J}{2\rho^3}\sin^2 I\cos 2\varphi$$

Recalling equation (6) for I, this leads to a splitting of the Hamiltonian into two parts:

$$\left.\begin{aligned}
H&=H_0+H_1\\
H_0&=\frac{1}{2}p_\rho^2+\frac{1}{2}\left(\frac{p_\varphi}{\rho}\right)^2-\frac{1}{\rho}+\frac{J}{2\rho^3}\left(\frac{1}{3}-\frac{p_\Omega^2}{p_\varphi^2}\right)\\
H_1&=-\frac{J}{2\rho^3}\sin^2 I\cos 2\varphi=\frac{J}{2\rho^3}\left(\frac{p_\Omega^2}{p_\varphi^2}-1\right)\cos 2\varphi
\end{aligned}\right\} \tag{11}$$

Thus, H_0 is independent of φ, while H_1 is periodic in φ with mean value equal to zero and vanishes for equatorial orbits.

It may be remarked that the Hamiltonian can always be split in this way for any disturbance function of the form of equation (9). The odd harmonics contribute nothing to H_0, since the odd powers of sin φ all have mean values equal to zero. The even harmonics contain only even powers of sin φ and hence contribute both to H_0 and H_1. For example, the contribution of the fourth harmonic to H_0 is

$$J_4\left(\frac{105}{64}\sin^4 I-\frac{15}{8}\sin^2 I+\frac{3}{8}\right)\Big/\rho^5$$

while its contribution to H_1 is

$$J_4\left[\left(\frac{15}{8}\sin^2 I-\frac{35}{16}\sin^4 I\right)\cos 2\varphi+\frac{35}{64}\sin^4 I\cos 4\varphi\right]\Big/\rho^5$$

For the main problem considered here these higher harmonics will be neglected.

The "intermediate" orbit mentioned in the introduction is defined as the solution of the problem when H_1 and the dissipative forces are neglected. This will be referred to as the "intermediate problem," and its solution is the primary objective of this report.

THE INTERMEDIATE PROBLEM

Substituting H_0 in equations (8) and dropping H_1 and D gives the intermediate problem

$$\left.\begin{aligned}
&\frac{d\rho}{d\tau}=p_\rho\\
&\frac{d\varphi}{d\tau}=\frac{p_\varphi}{\rho^2}+\frac{Jp_\Omega^2}{\rho^3 p_\varphi^3}\\
&\frac{d\Omega}{d\tau}=-\frac{Jp_\Omega}{\rho^3 p_\varphi^2}\\
&\frac{dp_\rho}{d\tau}=\frac{p_\varphi^2}{\rho^3}-\frac{1}{\rho^2}-\frac{J}{\rho^4}\left(1-\frac{3}{2}\sin^2 I\right)\\
&\frac{dp_\varphi}{d\tau}=0\\
&\frac{dp_\Omega}{d\tau}=0
\end{aligned}\right\}\tag{12}$$

with

$$\cos I=p_\Omega/p_\varphi \tag{13}$$

The last two of equations (12) show immediately that p_φ, p_Ω, and, hence, I are constants for the intermediate problem. In terms of physical quantities,

$$\left.\begin{aligned}
p_\varphi&=\rho W_\varphi\\
p_\Omega&=\rho W_\varphi\cos I\\
W_\varphi&=V_\varphi\sqrt{R/\mu}
\end{aligned}\right\}\tag{14}$$

and V_φ is the horizontal component of velocity. Thus, p_φ is the angular momentum, and p_Ω is the component of angular momentum along the polar axis of the planet (both in dimensionless form).

One method of attacking the intermediate problem (eqs. (12)) is to seek a solution of the Hamilton-Jacobi equation (ref. 7, ch. 8)

$$\frac{\partial \mathbf{F}}{\partial\tau}+H_0=0 \tag{15}$$

where

$$\mathbf{F}=\mathbf{F}(\tau,\rho,\varphi,\Omega,P_\rho,P_\varphi,P_\Omega)$$

is a function containing three arbitrary constants, P_ρ, P_φ, P_Ω, and in H_0 the momenta are to be replaced by:

$$\left.\begin{aligned}
p_\rho&=\partial\mathbf{F}/\partial\rho\\
p_\varphi&=\partial\mathbf{F}/\partial\varphi\\
p_\Omega&=\partial\mathbf{F}/\partial\Omega
\end{aligned}\right\}\tag{16}$$

If such a function, $\mathbf{F}$, can be found, then the complete solution of the intermediate problem is obtained by setting

$$\left.\begin{aligned}
\partial\mathbf{F}/\partial P_\rho&=Q_\rho\\
\partial\mathbf{F}/\partial P_\varphi&=Q_\varphi\\
\partial\mathbf{F}/\partial P_\Omega&=Q_\Omega
\end{aligned}\right\}\tag{17}$$

where, Q_ρ, Q_φ, Q_Ω are three additional arbitrary constants, giving six in all. In principle, the six equations, (16) and (17), can be solved for the six variables ρ, φ, Ω, p_ρ, p_φ, p_Ω in terms of τ and the six arbitrary constants. In practice, as will be seen below, the coordinates φ and Ω will be obtained directly from equations (12).

One method of solving the Hamilton-Jacobi equation (15) is by separating the variables, that is, by seeking a solution of the form

$$\mathbf{F}=F(\rho)+P_\varphi\varphi+P_\Omega\Omega-P_\rho\tau \tag{18}$$

where P_ρ, P_φ, P_Ω are arbitrary constants. This reduces equations (16) to:

$$\left.\begin{aligned} p_\rho &= \frac{dF}{d\rho} \\ p_\varphi &= P_\varphi \\ p_\Omega &= P_\Omega \end{aligned}\right\} \tag{19}$$

The Hamiltonian, H_0, equation (11), now becomes

$$H_0=\frac{1}{2}\left(\frac{dF}{d\rho}\right)^2+\frac{1}{2}\left(\frac{P_\varphi}{\rho}\right)^2-\frac{1}{\rho}-\frac{\sigma}{3\rho^3}$$

where

$$\sigma=-J\left(\frac{1}{2}-\frac{3}{2}\frac{P_\Omega^2}{P_\varphi^2}\right)=J\left(1-\frac{3}{2}\sin^2 I\right) \tag{20}$$

and the Hamilton-Jacobi equation (15) reduces to the ordinary differential equation

$$\frac{1}{2}\left(\frac{dF}{d\rho}\right)^2=P_\rho+\frac{1}{\rho}-\frac{1}{2}\left(\frac{P_\varphi}{\rho}\right)^2+\frac{\sigma}{3\rho^3} \tag{21}$$

In principle equation (21) can be integrated (as an elliptic integral of the third kind) to give $F(\rho)$. As is shown in appendix B, the six arbitrary constants then have the following physical significance:

P_ρ	total energy
P_φ	total angular momentum
P_Ω	polar component of angular momentum
$-Q_\rho$	time of perigee passage
Q_φ	argument of perigee
Q_Ω	right ascension of the ascending node when the satellite is at perigee

Equations (12) could then be integrated to give τ, φ, and Ω as elliptic integrals of the third kind in terms of ρ and the six constants, with the momenta being given directly by equations (19).

The transformation from the variables (ρ, φ, Ω, p_ρ, p_φ, p_Ω) to the set (Q_ρ, Q_φ, Q_Ω, P_ρ, P_φ, P_Ω) is, of course, a contact transformation. When H_1 and the dissipative forces are included, then the P's and Q's are no longer constants but satisfy a new set of canonical equations. This set of equations is exhibited in appendix C, to show explicitly the relation of the intermediate orbit to the complete problem; no attempt will be made to solve these equations in the present report.

To return to the intermediate problem, the method of solution outlined above is unsatisfactory for two reasons. Elliptic integrals of the third kind are notoriously intractable to analysis, either algebraic or numerical. Furthermore, in this representation all the coordinates, including time, are expressed as functions of ρ, whereas a solution with either τ or φ as the independent variable would be easier to interpret geometrically. While this ideal goal appears unattainable, it can be approached by introducing certain auxiliary variables, as will be seen in the next section. It then becomes possible to classify intermediate orbits according to a simple geometrical scheme, with energy and angular momentum as parameters. Closed form solutions will be obtained in terms of trigonometric, hyperbolic, or elliptic functions. The latter will be expanded into rapidly converging series, and natural generalizations of many classical equations and concepts will be exhibited.

While the main emphasis is on satellite orbits, all possible types of orbits will be exhibited and identified.

THE ENERGY EQUATION

If p_ρ is eliminated from equations (12), the resulting system is

$$\left.\begin{aligned} \frac{d^2\rho}{d\tau^2} &= \frac{P_\varphi^2}{\rho^3}-\frac{1}{\rho^2}-\frac{\sigma}{\rho^4} \\ \frac{d\varphi}{d\tau} &= \frac{P_\varphi}{\rho^2}+\frac{JP_\Omega^2}{P_\varphi^3\rho^3} \\ \frac{d\Omega}{d\tau} &= \frac{-JP_\Omega}{P_\varphi^2\rho^3} \end{aligned}\right\} \tag{22}$$

with σ given by equation (20) and

$$p_\varphi=P_\varphi$$

$$p_\Omega=P_\Omega$$

$$\cos I=\frac{P_\Omega}{P_\varphi}$$

P_φ and P_Ω being constants. Substituting the first of equations (19) into equation (21) gives

$$\frac{1}{2}\left(\frac{d\rho}{d\tau}\right)^2=P_\rho+\frac{1}{\rho}-\frac{1}{2}\left(\frac{P_\varphi}{\rho}\right)^2+\frac{\sigma}{3\rho^3} \tag{23}$$

which is simply the first integral (the energy integral) of the first of equations (22).

In the case of the classical two-body problem, with $J=\sigma=0$, the usual procedure is to take φ as the independent variable and to introduce a new dependent variable, u, which is the reciprocal of ρ:

$$u=\frac{1}{\rho} \tag{24}$$

In the present case equation (24) will still be used, but for the independent variable it is more convenient to introduce an angle, θ, defined (except for an additive constant) by

$$\frac{d\theta}{d\tau}=\frac{W_\varphi}{\rho}=\frac{P_\varphi}{\rho^2}=P_\varphi u^2 \tag{25}$$

Then

$$\left.\begin{aligned}\frac{d\rho}{d\tau}&=-P_\varphi\frac{du}{d\theta}\\ \frac{d^2\rho}{d\tau^2}&=-P_\varphi{}^2u^2\frac{d^2u}{d\theta^2}\end{aligned}\right\} \tag{26}$$

and equations (22) become

$$\left.\begin{aligned}2\xi\frac{d^2u}{d\theta^2}&=1-2\xi u+\sigma u^2\\ \frac{d\varphi}{d\theta}&=1+\frac{JP_\Omega{}^2}{P_\varphi{}^4}u\\ \frac{d\Omega}{d\theta}&=\frac{-JP_\Omega}{P_\varphi{}^3}u\end{aligned}\right\} \tag{27}$$

where

$$\xi=\frac{1}{2}P_\varphi^2 \tag{28}$$

and the energy equation (23) becomes

$$\xi\left(\frac{du}{d\theta}\right)^2=\eta+u-\xi u^2+\frac{1}{3}\sigma u^3 \tag{29}$$

where, for convenience in writing, the total energy is denoted by η:

$$\eta=P_\rho=\frac{1}{2}W^2-u-\frac{1}{3}\sigma u^3 \tag{30}$$

Thus the intermediate problem is reduced to solving the energy equation (29); once u has been obtained explicitly as a function of θ, the other quantities can be obtained from equations (25) and (27) by quadrature.

If higher harmonics are retained in the disturbance function, the energy equation becomes

$$\xi\left(\frac{du}{d\theta}\right)^2=\eta+u-\xi u^2+\frac{1}{3}\sigma u^3+\frac{1}{5}\sigma_4u^5\ldots$$

$$\sigma_4=J_4\left(\frac{105}{64}\sin^4 I-\frac{15}{8}\sin^2 I+\frac{3}{8}\right)$$

there being one additional term for each even harmonic.

It may be noticed in passing that the first of equations (27) has other physical applications. For example, if u is interpreted as a distance and θ as time, this equation describes the motion of an undamped spring with a nonlinear restoring force. The same equation also occurs in orbit theory when relativistic effects are included (ref. 8).

The parameters J and σ will not be permitted to assume arbitrary values. For the earth, $J\approx 0.0016$, and comparable values seem to be appropriate for the other planets. (An exhaustive study of the physical meaning of J is given by Jeffreys in ref. 9.) In this report the extremely conservative condition $J<1$ will be imposed, so that $-1/2<\sigma<1$. The analysis is simplified thereby, and no planet of the solar system will be excluded. This does, however, limit the generality of the analysis with respect to other physical applications.

It may also be remarked that in general the angle, θ, does not have any obvious geometrical significance. Qualitatively it differs from the argument of latitude, φ, by a small quantity of order J. However, in the case of polar orbits, $\theta=\varphi=\delta$ (except for an additive constant). In the case of equatorial orbits $\theta=\alpha$, and Ω is undefined and unneeded. Thus in the two extreme cases θ does have a simple interpretation—declination or right ascension.

To return to the analysis, the energy equation (29) is integrable in general in terms of elliptic functions and integrals. The precise nature of the solution depends on the roots of the "characteristic equation"

$$h(u)\equiv\eta+u-\xi u^2+\frac{1}{3}\sigma u^3=0 \tag{31}$$

Thus it is necessary to begin by studying this cubic equation.

THE CHARACTERISTIC EQUATION

The usual method of studying a cubic equation is known as Cardan's method (ref. 10, ch. IV). Making the substitution

$$w = \xi - \sigma u \tag{32}$$

gives the characteristic equation in the form

$$g(w) \equiv -\frac{1}{3}\,[w^3 + 3(\sigma - \xi^2)w + 2\xi^3 - 3\sigma\xi - 3\sigma^2\eta] = 0 \tag{33}$$

and the energy equation (29) becomes

$$\xi\left(\frac{dw}{d\theta}\right)^2 = g(w) \tag{34}$$

The discriminant of $g(w)$ is 27Δ, where

$$\Delta = 4(\xi^2 - \sigma)^3 - (3\sigma^2\eta + 3\sigma\xi - 2\xi^3)^2 \tag{35}$$

If $\xi^2 > \sigma$, Δ can be factored

$$\Delta = -9\sigma^4(\eta - \eta_1)(\eta - \eta_2) \tag{36}$$

where

$$\left.\begin{aligned} \eta_1 &= \frac{2\xi^3 - 3\sigma\xi + 2(\xi^2 - \sigma)^{3/2}}{3\sigma^2} \\ \eta_2 &= \frac{2\xi^3 - 3\sigma\xi - 2(\xi^2 - \sigma)^{3/2}}{3\sigma^2} \end{aligned}\right\} \tag{37}$$

The curves $\eta = \eta_1$, and $\eta = \eta_2$ are shown in figures 2 and 3 for $\sigma > 0$ and $\sigma < 0$, respectively (for example, for equatorial orbits, $\sigma = J$; for polar orbits, $\sigma = -J/2$). The significance of these curves is the following: If $\xi^2 > \sigma$ and $\eta_2 < \eta < \eta_1$, then the characteristic equation has three distinct real roots. If $\eta = \eta_1$ or η_2, then the characteristic equation has a double root. In all other cases there is one real root and a pair of conjugate complex roots.

Another important "curve" is the straight line

$$\eta = \eta_T \equiv \xi - 1 - \frac{1}{3}\sigma \tag{38}$$

along which $u = 1$ is one of the roots of the characteristic equation. This line is tangent to the curve $\eta = \eta_2$ at the point

$$\xi = \frac{1+\sigma}{2}, \qquad \eta = -\frac{1}{2} + \frac{1}{6}\sigma$$

(see figs. 2 and 3).

All possible intermediate orbits can now be classified as follows:

- $\eta \geq 0$ escape orbits
- $\eta < 0$ bounded orbits
- $\eta \geq \eta_T$ captive orbits (intersect the planet's surface)
- $\eta < \eta_T$ free orbits (do not intersect the planet's surface)

Free orbits can be further classified:

$\eta < \eta_T,\ \xi \leq \dfrac{1+\sigma}{2}$ entirely inside the planet

$\eta_2 \leq \eta < \eta_T,\ \xi > \dfrac{1+\sigma}{2}$ entirely outside the planet

Also, bounded, captive orbits will be called missile orbits, and, clearly, the satellite orbits are those that are bounded, free, and outside the planet.

These classifications are indicated on figures 2 and 3. Their proof involves actually obtaining the roots of the characteristic equation in every case and examining their behavior as functions of ξ and η. Only satellite orbits will be considered in detail, but appendix D contains a complete catalog of all possible orbits, with a brief discussion of each type.

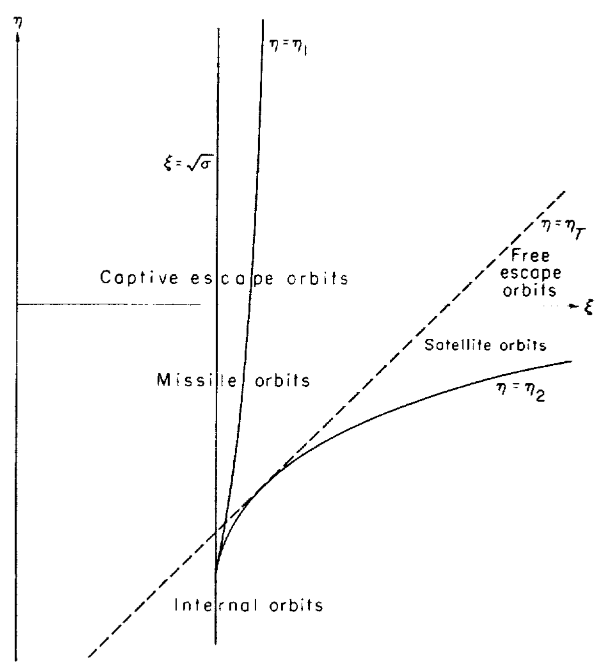

FIGURE 2.—Classification of orbits. Equatorial case: $\sigma > 0$.

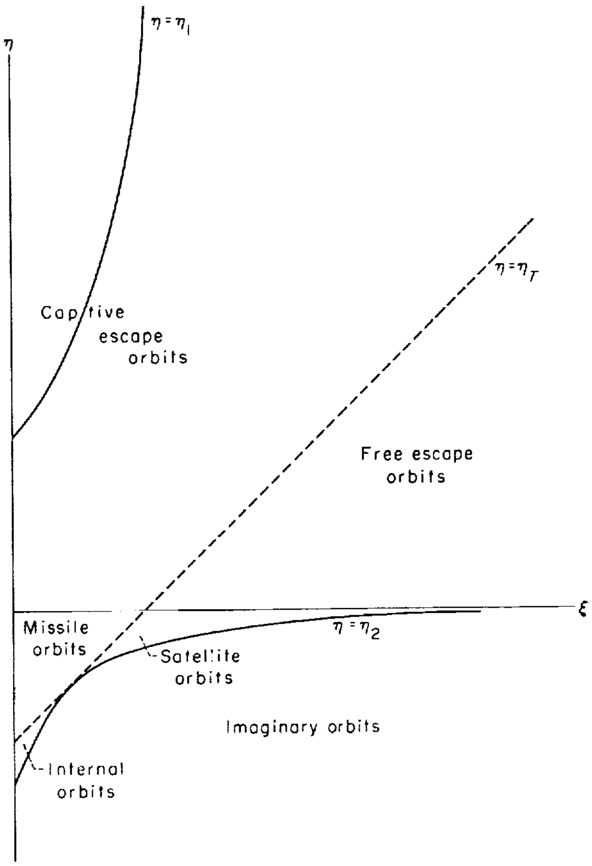

FIGURE 3.—Classification of orbits. Polar case: $\sigma<0$.

SATELLITE ORBITS

Satellite orbits are defined by the inequalities

$$\left.\begin{aligned}&\eta<0\\&\eta<\eta_T\\&\eta\geq\eta_2\\&\xi>\frac{1+\sigma}{2}\end{aligned}\right\}\tag{39}$$

However, the following analysis is applicable essentially to the entire region with a positive discriminant:

$$\xi^2>\sigma$$
$$\eta_2\leq\eta\leq\eta_1$$

Whenever the analysis is restricted, explicit mention will be made of the fact.

To solve the characteristic equation, define an angle ζ by

$$\sin^2\frac{3\zeta}{2}=\frac{\eta-\eta_2}{\eta_1-\eta_2},\ 0\leq\zeta\leq\frac{\pi}{3}\tag{40}$$

so that

$$\cos 3\zeta=\frac{2\xi^3-3\sigma\xi-3\sigma^2\eta}{2(\xi^2-\sigma)^{3/2}}\tag{41}$$

and the characteristic equation becomes

$$g(w)=-\frac{1}{3}\,[w^3-3(\xi^2-\sigma)w+2(\xi^2-\sigma)^{3/2}\cos 3\zeta]=0\tag{42}$$

The three real roots are (ref. 10, ch. IV):

$$\left.\begin{aligned}&w_1=2\sqrt{\xi^2-\sigma}\cos\left(\frac{\pi}{3}-\zeta\right)\\&w_2=2\sqrt{\xi^2-\sigma}\cos\left(\frac{\pi}{3}+\zeta\right)\\&w_3=-2\sqrt{\xi^2-\sigma}\cos\zeta\\&u_i=\frac{\xi-w_i}{\sigma},\ i=1,2,3\end{aligned}\right\}\tag{43}$$

and clearly

$$\left.\begin{aligned}&w_1>w_2>w_3\\&w_1>0\\&w_3<0\\&w_1+w_2+w_3=0\end{aligned}\right\}\tag{44}$$

The energy equation becomes

$$3\xi\left(\frac{dw}{d\theta}\right)^2=-(w-w_1)(w-w_2)(w-w_3)$$

and there are two solutions (ref. 11, eqs. 232.00 and 236.00):

$$w=w_3-(w_2-w_3)\ tn\ ^2(f\theta+C)\tag{45}$$

and

$$\left.\begin{aligned}w&=w_1+(w_2-w_1)sn^2(f\theta+C)\\&=w_2+(w_1-w_2)cn^2(f\theta+C)\end{aligned}\right\}\tag{46}$$

where tn, sn, cn are elliptic functions with modulus k, and

$$\left.\begin{aligned}&k^2=\frac{w_1-w_2}{w_1-w_3}=\frac{\sin\zeta}{\sin\left(\zeta+\frac{\pi}{3}\right)}\\&4f^2=\frac{w_1-w_3}{3\xi}=\frac{\sin\left(\zeta+\frac{\pi}{3}\right)}{2\xi}\sqrt{\frac{\xi^2-\sigma}{3}}\end{aligned}\right\}\tag{47}$$

and C is a constant to be specified later.

The first solution, equation (45), is physically unrealizable, since, if $\sigma>0$,

$$u=\frac{\xi-w}{\sigma}>\frac{\xi-w_3}{\sigma}>\frac{\xi}{\sigma}>\frac{1}{\sqrt{\sigma}}>1, \quad \rho<1$$

while, if $\sigma<0$,

$$u=\frac{\xi-w}{\sigma}<\frac{\xi-w_3}{\sigma}<0, \quad \rho<0$$

that is, if $\sigma>0$, the orbit lies entirely inside the planet, while, if $\sigma<0$, the orbit is imaginary, since ρ cannot be negative.

In terms of u, the second solution, equation (46) becomes

$$\left.\begin{aligned} u&=u_1+(u_2-u_1)\ sn^2\ (f\theta+C) \\ &=u_2+(u_1-u_2)\ cn^2\ (f\theta+C) \end{aligned}\right\} \tag{48}$$

TRANSFORMATION TO CLASSICAL FORM

For mnemonic purposes, use subscript p for perigee and a for apogee. Then

$$\left.\begin{aligned} &\left.\begin{aligned} u_p&=u_2 \\ u_a&=u_1 \end{aligned}\right\} \text{ if } \sigma>0 \\ &\left.\begin{aligned} u_p&=u_1 \\ u_a&=u_2 \end{aligned}\right\} \text{ if } \sigma<0 \end{aligned}\right\} \tag{49}$$

and, by equation (48), $u_a\leq u\leq u_p$ in either case. Equations (43) can be used directly to show that $u_a=0$ when $\eta=0$, and, on the line $\eta=\eta_T$,

$$u_p=1 \text{ if } \xi>\frac{1+\sigma}{2}$$

$$u_a=1 \text{ if } \xi<\frac{1+\sigma}{2}$$

It then follows from equations (40) and (43) that, if ξ is held fixed and η is increased, then the perigee distance $(\rho_p=1/u_p)$ decreases and the apogee distance $(\rho_a=1/u_a)$ increases. This is valid in the entire region where the discriminant is positive $(\xi^2>\sigma,\ \eta_2<\eta<\eta_1)$ and is sufficient to prove the classifications mentioned earlier and indicated in figures 2 and 3.

For free orbits $(\xi>(1+\sigma)/2,\ \eta<\eta_T)$ it is convenient to choose the constant, C, of equation (48) equal to K (the complete elliptic integral of the first kind) if $\sigma>0$ and equal to zero if $\sigma<0$. Equation (48) then becomes, by virtue of equations (49),

$$\left.\begin{aligned} u&=u_a+(u_p-u_a)\ cd^2 f\theta \quad \text{if } \sigma>0 \\ u&=u_a+(u_p-u_a)\ cn^2 f\theta \quad \text{if } \sigma<0 \end{aligned}\right\} \tag{50}$$

In either case, $\theta=0$ at perigee and $\theta=K/f$ at apogee (of course, this value is never attained for escape orbits).

Since the nature of the elliptic functions cn and cd is strongly dependent on the magnitude of the modulus, k, it is desirable to estimate this quantity. Recalling that ζ is a function of ξ and η, it can be shown that, for free orbits $(\xi>(1+\sigma)/2,\ \eta<\eta_T)$, ζ has a maximum at $\xi=1$, $\eta=\eta_T$. Evaluating ζ at this point gives the upper bounds

$$\zeta\leq\frac{1}{3}\text{ arc sin}\left|\frac{\sigma}{2}\sqrt{3+\sigma}\right|$$

For the earth, $\sigma\approx0.0016$, giving, by means of equation (47),

$$k^2<6\times10^{-4}$$
$$q<4\times10^{-5}$$

where q is Jacobi's nome (ref. 11, eq. 901.00). As will be seen later, these bounds permit the expansion of the solution in rapidly converging series. In particular, considerable use will be made of the Fourier series (ref. 12, p. 520, example 5)

$$sn^2 f\theta=\frac{K-E}{Kk^2}-\frac{2\pi^2}{K^2k^2}\sum_{m=1}^{\infty}\frac{mq^m}{1-q^{2m}}\cos\frac{m\pi f}{K}\theta \tag{51}$$

and the relations (ref. 11, eqs. 121.00 and 122.03)

$$\left.\begin{aligned} cn^2 f\theta&=1-sn^2 f\theta \\ cd^2 f\theta&=sn^2\ (f\theta+K) \end{aligned}\right\} \tag{52}$$

where E is the complete elliptic integral of the second kind.

Equations (50) can be transformed into the classical two-body form by generalizing the concepts of semilatus rectum (p), eccentricity (ϵ), semimajor axis (a), true anomaly (v), and eccentric anomaly (E) as follows:

$$\left.\begin{aligned} p&=\frac{2}{u_p+u_a} \\ \epsilon&=p\frac{u_p-u_a}{2} \\ a&=\frac{p}{1-\epsilon^2} \\ &\left.\begin{aligned}\cos(v/2)&=cd f\theta\\ \sin(v/2)&=k' sd f\theta\end{aligned}\right\}\text{ if } \sigma>0 \\ &\left.\begin{aligned}\cos(v/2)&=cn f\theta\\ \sin(v/2)&=sn f\theta\end{aligned}\right\}\text{ if } \sigma<0 \\ k'&=\sqrt{1-k^2} \\ \tan\frac{E}{2}&=\sqrt{\frac{1-\epsilon}{1+\epsilon}}\tan\frac{v}{2},\ (\epsilon<1)\end{aligned}\right\} \tag{53}$$

All the classical relations between p, ϵ, a, v, E, ρ_p, ρ_a are thus preserved, as well as the representation of the orbit as a conic in the polar coordinates ρ, v:

$$\left.\begin{aligned} \rho&=a\,(1-\epsilon\cos E)=\frac{p}{1+\epsilon\cos v} \\ \rho_p&=a\,(1-\epsilon)=\frac{p}{1+\epsilon} \\ \rho_a&=a\,(1+\epsilon)=\frac{p}{1-\epsilon} \\ \sin E&=\sqrt{1-\epsilon^2}\,\frac{\sin v}{1+\epsilon\cos v} \\ \cos E&=\frac{\epsilon+\cos v}{1+\epsilon\cos v} \\ \sin v&=\sqrt{1-\epsilon^2}\,\frac{\sin E}{1-\epsilon\cos E} \\ \cos v&=\frac{\cos E-\epsilon}{1-\epsilon\cos E} \\ v-E&=2\text{ arc tan}\left(\frac{\epsilon\sin v}{1+\epsilon\cos v+\sqrt{1-\epsilon^2}}\right) \\ &=2\text{ arc tan}\left(\frac{\epsilon\sin E}{1-\epsilon\cos E+\sqrt{1-\epsilon^2}}\right) \\ \frac{dv}{dE}&=\frac{\sqrt{1-\epsilon^2}}{1-\epsilon\cos E}=\frac{1+\epsilon\cos v}{\sqrt{1-\epsilon^2}}\end{aligned}\right\} \tag{54}$$

Note first that ϵ does have the usual properties associated with the term "eccentricity" (see also eqs. (56) and (57) below):

$\epsilon=0$ if $\eta=\eta_2$ (circular orbit)

$\epsilon=1$ if $\eta=0$ (critical escape orbit)

$\epsilon>1$ if $\eta>0$ (supercritical escape orbit), for free orbits

If it is recalled that u_p and u_a both satisfy the characteristic equation

$$\eta+u_p-\xi u_p^2+\frac{1}{3}\sigma u_p^3=0$$

$$\eta+u_a-\xi u_a^2+\frac{1}{3}\sigma u_a^3=0$$

it is possible to express ξ and η as functions of p, a, and ϵ:

$$\left.\begin{aligned}\xi&=\frac{p}{2}+\frac{\sigma(3+\epsilon^2)}{6p}\\ \eta&=-\frac{1}{2a}+\frac{\sigma}{6\,pa^2}\end{aligned}\right\} \tag{55}$$

While every quantity is computable from the equations given, their nature is clarified by expanding them as power series in σ, the leading term being in each case the ordinary "two-body" value. All the series given here are derived in detail in appendix E. Abbreviated versions are listed here in order to show their qualitative behavior. First define ϵ_0 and λ by

$$\left.\begin{aligned}\epsilon_0&=\sqrt{4\xi(\eta-\eta_2)}\\ \lambda&=\sigma/\xi^2\end{aligned}\right\} \tag{56}$$

Then the parameters ϵ, p, a are given by

$$\left.\begin{aligned}\epsilon&=\epsilon_0\left[1-\frac{1+4\xi\eta}{12}\lambda-\left(\frac{1}{48}+\frac{\xi\eta}{8}-\frac{\xi^2\eta^2}{9}\right)\lambda^2\ldots\right]\\ \epsilon^2&=1+4\xi\eta-4\xi\eta\,\frac{1+2\xi\eta}{3}\lambda-4\xi\eta\,\frac{1+\xi\eta-3\xi^2\eta^2}{9}\lambda^2\ldots\\ p&=2\xi\left(1-\frac{1+\xi\eta}{3}\lambda-\frac{1+\xi\eta-\xi^2\eta^2}{9}\lambda^2\ldots\right)\\ a&=\frac{-1}{2\eta}\left(1+\frac{\xi\eta}{3}\lambda+\frac{\xi\eta}{9}\lambda^2\ldots\right)\end{aligned}\right\} \tag{57}$$

If it is recalled that 2ξ is the square of the angular momentum and η is the total energy, it is clear that these do reduce to the two-body values when $J=\sigma=0$.

It should be emphasized that the orbit is a conic only in the plane of (ρ, v). In the physical plane (ρ, θ), for equatorial and polar orbits, or (ρ, φ) for inclined orbits, the curve is a distorted conic, the difference between θ (or φ), and v being a measure of the distortion. The nature of this distortion is clarified by a study of the transformation between v and θ (eqs. (53));

$$\left.\begin{aligned}\cos(v/2)&=cdf\theta\\ \sin(v/2)&=k'\,sdf\theta\end{aligned}\right\}\text{ if }\sigma>0$$

$$\left.\begin{aligned}\cos(v/2)&=cnf\theta\\ \sin(v/2)&=snf\theta\end{aligned}\right\}\text{ if }\sigma<0$$

Differentiating (by means of ref. 11, eqs. 731.01, 731.11, and 121.00)

$$\left.\begin{aligned}&\left.\begin{aligned}\frac{d\theta}{dv}&=\frac{1}{2f\sqrt{1-k^2\cos^2(v/2)}}\\ \frac{dv}{d\theta}&=2fk'ndf\theta\end{aligned}\right\}\text{ if }\sigma>0\\ &\left.\begin{aligned}\frac{d\theta}{dv}&=\frac{1}{2f\sqrt{1-k^2\sin^2(v/2)}}\\ \frac{dv}{d\theta}&=2fdnf\theta\end{aligned}\right\}\text{ if }\sigma>0\end{aligned}\right\}\tag{58}$$

As is shown in appendix E, these equations lead to the same formal series connecting v and θ, independently of the sign of σ:

$$\left.\begin{aligned}\theta&=\frac{K}{\pi f}v+\sum_{m=1}^{\infty}\frac{B_m}{m}\sin mv\\ v&=\frac{\pi f}{K}\theta-\sum_{m=1}^{\infty}\theta_m\sin\frac{m\pi f}{K}\theta\end{aligned}\right\}\tag{59}$$

where

$$\left.\begin{aligned}B_1&=\frac{\epsilon_0}{12}\lambda+\frac{\epsilon_0}{12}\lambda^2\ldots\\ \frac{B_2}{2}&=\frac{\epsilon_0^2}{384}\lambda^2\ldots\\ \theta_1&=\frac{\epsilon_0}{12}\lambda+\frac{\epsilon_0}{16}\lambda^2\ldots\\ \theta_2&=-\frac{\epsilon_0^2}{1152}\lambda^2\ldots\end{aligned}\right\}\tag{60}$$

and the secular coefficients are

$$\left.\begin{aligned}\frac{K}{\pi f}&=1+\frac{1}{4}\lambda+\left(\frac{35}{192}+\frac{5\xi\eta}{48}\right)\lambda^2\ldots\\ \frac{\pi f}{K}&=1-\frac{1}{4}\lambda-\left(\frac{23}{192}+\frac{5\xi\eta}{48}\right)\lambda^2\ldots\end{aligned}\right\}\tag{61}$$

The geometrical character of the orbit is now clearly shown by the first of equations (54)

$$\rho=\frac{p}{1+\epsilon\cos v}$$

and equations (59). The orbit is a conic section in the (ρ, v) plane (ellipse, parabola, or hyperbola according as $\epsilon<1$, $=1$, or >1). Equation (59) simply represents a rotation of each point by the amount $\theta-v$. This rotation consists of two parts: a secular part, that is linear in v, and a harmonic part, of period 2π in v, that vanishes at every perigee and apogee. Because of the secular motion successive perigees do not coincide in their values of θ (reduced modulo 2π). Instead, each perigee is in "advance" of its predecessor by the amount

$$\Delta\theta=\frac{\pi\sigma}{2\xi^2}\left[1+\left(\frac{35}{48}+\frac{5\xi\eta}{12}\right)\lambda+\ldots\right]\tag{62}$$

To first order in J, this advance is

$$\Delta\theta\approx\frac{2\pi J\left(1-\frac{3}{2}\sin^2 I\right)}{p^2}\tag{63}$$

(Note that $\Delta\theta>0$ for equatorial orbits and $\Delta\theta<0$ for polar orbits.)

For true equatorial orbits ($I=0$), θ is the right ascension, and equation (62) gives the total advance of perigee. For inclined orbits, the angles φ and Ω must be studied in order to describe precisely the motion of the perigee and the node.

It is interesting to note for critical escape orbits, with $\eta=0$, $\epsilon=1$, that the asymptotic direction is

$$\theta=\pm\frac{K}{\pi f}\pi\text{ as }\rho\rightarrow\infty$$

For equatorial orbits, $K/\pi f>1$, so that the orbit crosses itself (see ref. 8).

MOTION OF THE NODE AND PERIGEE

Since u has been obtained as a simple function of the true anomaly, v, the other quantities can be obtained by quadrature directly from equations (25) and (27), combined with equation (E20). The integration is straightforward in the case of the node and perigee, and the results are:

$$\left.\begin{aligned} \Omega &= \Omega_p+\Omega_s v+\sum_{m=1}^{\infty}\Omega_m \sin mv \\ \varphi &= \varphi_p+\theta-\frac{P_\Omega}{P_\varphi}(\Omega-\Omega_p) \end{aligned}\right\} \tag{64}$$

or, in series form

$$\varphi=\varphi_p+\varphi_s v+\sum_{m=1}^{\infty}\varphi_m \sin mv \tag{65}$$

where the coefficients are

$$\left.\begin{aligned} \Omega_s &= -\frac{JP_\Omega}{P_\varphi^3 p}\left(\frac{K}{\pi f}+\frac{\epsilon}{2}B_1\right) \\ \Omega_1 &= -\frac{JP_\Omega}{P_\varphi^3 p}\left[B_1+\epsilon\left(\frac{K}{\pi f}+\frac{1}{2}B_2\right)\right] \\ \Omega_m &= -\frac{JP_\Omega}{P_\varphi^3 p}\,\frac{B_m+\frac{\epsilon}{2}(B_{m-1}+B_{m+1})}{m},\quad m=2,3,\ldots \end{aligned}\right\} \tag{66}$$

where the coefficients B_m are given in appendix E, equation (E19), and the coefficients for φ are

$$\left.\begin{aligned} \varphi_s &= \frac{K}{\pi f}-\frac{P_\Omega}{P_\varphi}\Omega_s \\ \varphi_m &= \frac{B_m}{m}-\frac{P_\Omega}{P_\varphi}\Omega_m,\quad m=1,2,3 \end{aligned}\right\} \tag{67}$$

Note that

$$\Omega_m=0(J^m)$$

$$\varphi_m=0(J^m)$$

so that these series converge rapidly.

The secular terms, Ω_s and φ_s, can be put in a more familiar form by inserting the series expansions for p, K, etc., and recalling that

$$\sigma=J\left(1-\frac{3}{2}\sin^2 I\right)$$

$$P_\Omega=P_\varphi \cos I$$

$$P_\varphi^2=2\xi$$

The results, to the second order in J, are

$$\Omega_s=-\frac{J\cos I}{4\xi^2}\left[1+J\,\frac{5+4\xi n}{8}\,\frac{\left(1-\frac{3}{2}\sin^2 I\right)}{\xi^2}+\cdots\right]$$

$$\varphi_s=1+J\,\frac{\left(2-\frac{5}{2}\sin^2 I\right)}{4\xi^2}+\frac{J^2\left(1-\frac{3}{2}\sin^2 I\right)}{4\xi^4}\left[\frac{83}{48}+\frac{5\xi n}{12}-\left(\frac{67}{32}+\frac{5\xi n}{5}\right)\sin^2 I\right]+\cdots$$

To the first order in J, if it is recalled that

$$2\xi=p+O(J)$$

the secular terms are

$$\Omega_s=-\frac{J\cos I}{p^2}+\cdots$$

$$\varphi_s=1+\frac{J\left(2-\frac{5}{2}\sin^2 I\right)}{p^2}+\cdots$$

in complete agreement with the results of reference 1.

Thus, for inclined orbits, the advance of the node in one period is

$$\Delta\Omega=-\frac{2\pi J\cos I}{p^2}+O(J^2)$$

and the advance of perigee is (when reduced modulo 2π)

$$\Delta\varphi=\frac{\pi J(4-5\sin^2 I)}{p^2}+O(J^2)$$

Equation (65) can be inverted to give the true anomaly, v, explicitly in terms of φ (see appendix E).

$$v=\frac{\varphi-\varphi_p}{\varphi_s}+\sum_{m=1}^{\infty} v_m \sin m \frac{\varphi-\varphi_p}{\varphi_s} \tag{68}$$

To second order in J the coefficients are

$$\left.\begin{aligned} v_1&=-\frac{\varphi_1}{\varphi_s} \\ v_2&=-\frac{\varphi_2}{\varphi_s}+\frac{1}{2}\frac{\varphi_1^2}{\varphi_s^2} \end{aligned}\right\} \tag{69}$$

Since the equation of the orbit is

$$\rho=\frac{p}{1+\epsilon \cos v}$$

it is convenient to have a series for $\cos v$ in terms of φ (see appendix E):

$$\cos v=\sum_{m=0}^{\infty} C_m \cos m \frac{\varphi-\varphi_p}{\varphi_s} \tag{70}$$

To second order in J the coefficients are

$$\left.\begin{aligned} C_0&=-\frac{1}{2} v_1 \\ C_1&=1-\frac{1}{8} v_1^2-\frac{1}{2} v_2 \\ C_2&=\frac{1}{2} v_1 \\ C_3&=\frac{1}{2} v_2 \end{aligned}\right\} \tag{71}$$

Kinematics: Kepler's Equation

The kinematics of the motion are given by equation (25)

$$\frac{d\theta}{d\tau}=P_\varphi u^2$$

Inserting u from equation (54) and $d\theta/dv$ from equation (58) gives

$$\frac{d\tau}{dv}=\frac{p^2}{2fP_\varphi}\frac{1}{(1+\epsilon \cos v)^2\sqrt{1-k^2\cos^2(v/2)}}, \quad \sigma>0$$

$$\frac{d\tau}{dv}=\frac{p^2}{2fP_\varphi}\frac{1}{(1+\epsilon \cos v)^2\sqrt{1-k^2\sin^2(v/2)}}, \quad \sigma<0$$

Expanding the radicals by the binomial theorem and integrating term-by-term gives the generalization of Kepler's equation

$$M\equiv n(\tau-\tau_p)=\sum_{m=0}^{\infty} M_m k^{2m} \tag{72}$$

where M can be called the mean anomaly and n the mean motion, with

$$n=a^{-3/2} \quad 2f\sqrt{\frac{2\xi}{p}} \tag{73}$$

and

$$M_m=\binom{2m}{m}\left(\frac{1}{4}\right)^m (1-\epsilon^2)^{3/2}\int_0^v \frac{\cos^{2m}(\psi/2)}{(1+\epsilon\cos\psi)^2}d\psi, \quad \sigma>0$$

$$M_m=\binom{2m}{m}\left(\frac{1}{4}\right)^m (1-\epsilon^2)^{3/2}\int_0^v \frac{\sin^{2m}(\psi/2)}{(1+\epsilon\cos\psi)^2}d\psi, \quad \sigma<0$$

These integrals are evaluated in appendix E; the first three are

$$\left.\begin{aligned} M_0&=E-\epsilon \sin E \\ M_1&=\frac{1\pm\epsilon}{4}(E\pm\sin E) \\ M_2&=\frac{3}{32\epsilon^2}\{(1-\epsilon^2)^{3/2}v-(1\mp\epsilon)^2[(1\mp 2\epsilon)E+\epsilon\sin E]\} \end{aligned}\right\} \tag{74}$$

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the upper sign to be used when $\sigma>0$, the lower when $\sigma<0$.

Note that equation (72) reduces to the classical form of Kepler's equation in the case of vanishing oblateness ($\sigma=0$, $k=0$, $p=2\xi$, $f=1/2$).

The analysis is now complete. The kinematical description of the orbit is given by equation (72), the geometrical description by equations (53), (54), (64), (65), (68), and (70). A numerical example of the use of these equations is presented in the next section.

NUMERICAL EXAMPLE

ORBIT DETERMINATION

As an illustration of the present theory, an intermediate orbit will be fitted to the satellite 1958 $\beta 2$ (Vanguard 1) for the epoch 02 November 1960, 1227 U. T. (ref. 13). The six given pieces of data are

Anomalistic period	134.03048 minutes
Inclination	34° .245
Right ascension of ascending node	131° .796
Argument of perigee	47° .691
Eccentricity	0.18977
Mean anomaly	222° .764

The conventional methods of celestial mechanics give an osculating ellipse with

Semimajor axis	1.3601810R
Semilatus rectum	1.3111973R

Solving Kepler's equation gives the position at epoch

$$\rho=1.5661320$$
$$\varphi=258°\ .6233$$

To fit an intermediate orbit to these data, the following quantities will be preserved

I	Inclination
Ω	Right ascension of ascending node
η	Energy
ξ	Angular momentum
ρ	Geocentric distance
φ	Argument of latitude

The quantities η and ξ are directly obtainable from the semimajor axis and the semilatus rectum by means of equation (55) with $\sigma=0$. Thus, the data are

Epoch	02 November 1960, 1227 U. T.
I	34° .245
Ω	131° .796
η	−0.36759813
ξ	0.65559865
ρ	1.5661320
φ	258° .6233

Straightforward application of the series expansions (appendix E) gives

$\sigma=0.000852176$	$B_2=7\times10^{-10}$
$\lambda=0.00198268$	$\Omega_s=-0.000781244$
$\epsilon_0=0.19063927$	$\Omega_1=-0.000148959$
$\epsilon=0.19063815$	$\Omega_2=1\times10^{-9}$
$a=1.3599642$	$\varphi_s=1.001142097$
$p=1.3105392$	$\varphi_1=0.000154694$
$2f=0.99953549$	$\varphi_2=1\times10^{-9}$
$n=0.63039973$	$v_1=-0.000154518$
$k^2=0.000126172$	$v_2=1\times10^{-8}$
$K/\pi f=1.00049629$	$C_0=0.00007726$
$A_1=0.0000315437$	$C_1=0.999999986$
$A_2=7\times10^{-10}$	$C_2=-0.00007726$
$B_1=0.0000315584$	$C_3=5\times10^{-9}$

Since the value of ρ at epoch is known, the value of the eccentric anomaly, E, is obtainable from

$$\rho=a(1-\epsilon\cos E)$$

the true anomaly, v, from

$$\rho=\frac{p}{1+\epsilon\cos v}$$

and the mean anomaly, M, from Kepler's equation. The values are

$$E=217°.3246$$

$$v=211°.1216$$

$$M=223°.9521$$

Since the values of Ω, φ, and v are known at epoch, the values of Ω_p and φ_p are obtainable from equations (64) and (65). Summarizing, the equations of the orbit are:

$$\left.\begin{aligned}
I&=34^\circ.245\\
\rho&=1.3599642(1-0.19063815\cos E)\\
&=\frac{1.3105392}{1+0.19063815\cos v}\\
\Omega&=131^\circ.9565-0.000148959\,v^\circ-0^\circ.00853\sin v\\
\varphi&=47^\circ.3064+1.001142097\,v^\circ+0^\circ.08863\sin v\\
v^\circ&=0.99885921(\varphi^\circ-47^\circ.3064)\\
&\quad-0^\circ.008853\sin[0.99885921(\varphi^\circ-47^\circ.3064)]\\
\cos v&=0.000077259\,\{1-\cos 2[0.99885921(\varphi^\circ-47^\circ.3064)]\}\\
&\quad+0.999999986\cos[0.99885921(\varphi^\circ-47^\circ.3064)]
\end{aligned}\right\}\tag{75}$$

Kepler's equation is

$$\left.\begin{aligned}
M^\circ&=223^\circ.952+2^\circ.6860248\,t(\text{minutes})\\
&=1.000025530\,E^\circ-10^\circ.92130\sin E
\end{aligned}\right\}\tag{76}$$

Anomalistic period = 134.03048 minutes

$$\left.\begin{aligned}
\sin E&=0.98166038\frac{\sin v}{1+0.19063815\cos v}\\
\cos E&=\frac{0.19063815+\cos v}{1+0.19063815\cos v}\\
\sin v&=0.98166038\frac{\sin E}{1-0.19063815\cos E}\\
\cos v&=\frac{\cos E-0.19063815}{1-0.19063815\cos E}
\end{aligned}\right\}\tag{77}$$

To obtain the velocity components, equations (14) and (28) give the horizontal component

$$V_\varphi=\sqrt{\frac{\mu}{R}}\frac{1}{\rho}\sqrt{2\xi}=\frac{1}{p}\sqrt{\frac{2\xi\mu}{R}}(1+\epsilon\cos v)$$

Thus, for this orbit,

$$V=15451.40(1+0.19063815\cos v)\quad\text{miles/hour}$$

To obtain the vertical component, equations (26) and (58) give

$$\frac{d\rho}{d\tau}=-P_\varphi 2f\sqrt{1-k^2\cos^2\frac{v}{2}}\,\frac{du}{dv}$$

Since $u=(1/p)\ (1+\epsilon\cos v)$, the vertical component is

$$\frac{dr}{dt}=\frac{2f\epsilon}{p}\sin v\sqrt{\frac{\mu}{R}\left(1-k^2\cos^2\frac{v}{2}\right)2\xi}$$

Thus, for this orbit,

$$\frac{dr}{dt}=2944.259\sin v\sqrt{1-0.000126172\cos^2\frac{v}{2}}\ \text{miles/hour}$$

The orbit is a spiral, with a fixed perigee and apogee for each revolution. The altitudes of perigee and apogee are 399.1 miles and 2454.2 miles, respectively. The speed at perigee is 18397.03 miles per hour, and the speed at apogee is 12505.77 miles per hour. Each perigee is 0°.411155 in advance of its predecessor, and each ascending node is 0°.28125 west of its predecessor (in right ascension).

EPHEMERIS COMPUTATION

Computation of position at specified times.—When the time is specified, the procedure is to compute the eccentric anomaly, E, from Kepler's equation (eq. (76)), the true anomaly, v, from equations (77), and the remaining quantities from equations (75). This has been done for various times, and the following table shows the comparison between predicted and observed positions (the "observed" positions having been computed from the data of ref. 13).

	Observed	Predicted
09 Nov. 1960, 1227 U. T.		
φ	354°.731	355°.596
Altitude	1128.9 miles	1109.2 miles
Ω	110°.630	110°.655
25 Nov. 1960, 1227 U. T.		
φ	31°.437	31°.282
Altitude	1735.5 miles	1737.2 miles
Ω	62°.245	62°.307

Computation of time of equator crossings.—To compute the time corresponding to a prescribed position, the procedure is to compute v from φ by equation (75), E from v by equation (77), and t from Kepler's equation. The following table shows the comparison between predicted and observed times and altitudes:

Pass no.	Date	Observed time			Predicted time			Observed altitude, km	Predicted altitude, km
		Hr	Min	Sec	Hr	Min	Sec		
10360	05 Nov. 1960	12	30	58	12	30	35	1282	1269
10361	05 Nov. 1960	14	44	53	14	44	30	1290	1278
10535	21 Nov. 1960	18	59	15	18	59	10	3219	3221
10651	02 Dec. 1960	13	42	06	13	42	18	3941	3950

ORBIT ACQUISITION

In this section the equations are displayed that provide the elements of an orbit when the position and velocity vectors are known at a given time. Specifically, let the position be specified by

r_0 geocentric distance
α_0 right ascension
δ_0 declination

Let the velocity vector, as seen by an observer on the earth, be specified by

V_0 speed relative to the earth
β_e azimuth, clockwise from north
γ_e flight path angle, upward from the horizontal

Denoting the earth's angular spin velocity by ω, the velocity components with respect to the inertial coordinate system are found to be

$$V_r = V_0 \sin \gamma_e$$
$$V_\alpha = V_0 \cos \gamma_e \sin \beta_e + r_0\omega \cos \delta_0$$
$$V_\delta = V_0 \cos \gamma_e \cos \beta_e$$

The horizontal component of the velocity with respect to the inertial coordinate system is

$$V_\varphi = \sqrt{V_\alpha^2 + V_\delta^2}$$

and its azimuth β, is given by

$$\cos \beta = \frac{V_\delta}{V_\varphi}, \qquad \sin \beta = \frac{V_\alpha}{V_\varphi}$$

The equations of appendix A then yield

$$\cos I = \cos \delta_0 \sin \beta$$
$$\sin (\alpha_0 - \Omega_0) = \tan \delta_0 \operatorname{ctn} I$$
$$\cos (\alpha_0 - \Omega_0) = \cos \beta / \sin I$$

which determine the inclination, I, and the right ascension of the ascending node, Ω_0. The dimensionless geocentric distance is

$$\rho_0 = r_0/R$$
$$u_0 = 1/\rho_0$$

and the dimensionless velocity components are

$$W_\varphi = V_\varphi \sqrt{R/\mu}, \qquad W_\rho = V_r \sqrt{R/\mu}$$

The parameter ξ is given by equations (28) and (14)

$$\xi = \frac{1}{2} (\rho_0 W_\varphi)^2$$

and the dimensionless energy, η, is given by equations (30) and (20)

$$\eta = \frac{1}{2} (W_\varphi^2 + W_\rho^2) - u_0 - \frac{1}{3} \sigma u_0^3$$

$$\sigma = J\left(1 - \frac{3}{2} \sin^2 I\right)$$

The various parameters a, p, ϵ, f, etc., are then obtained from ξ and η as in the preceding section, by means of the series expansions of appendix E.

The initial value of the argument of latitude, φ_0, is given by (see appendix A)

$$\sin \varphi_0 = \frac{\sin \delta_0}{\sin I}$$

$$\cos \varphi_0 = \cos \delta_0 \cos (\alpha_0 - \Omega_0)$$

The initial value of the true anomaly, v_0, is given by

$$\rho_0 = \frac{p}{1+\epsilon \cos v_0}$$

To determine the proper quadrant note that

$$W_\rho = \frac{d\rho}{d\tau} = -\frac{1}{u^2}\frac{du}{d\theta}\frac{d\theta}{d\tau} = -\frac{1}{u^2}\frac{du}{dv}\frac{dv}{d\theta}\frac{d\theta}{d\tau} = \frac{\epsilon \sin v}{u^2 p}\frac{dv}{d\theta}\frac{d\theta}{d\tau}$$

so that W_ρ and $\sin v$ have the same sign.

The eccentric anomaly, E_0, is given by equations (54), and the mean anomaly, M_0, by Kepler's equation. Finally, with Ω, φ, and v known, the remaining constants Ω_p and φ_p are given by equations (64) and (65).

RELATIVISTIC EFFECTS

It was mentioned earlier that the energy equation (eq. (29)) has an application in the general theory of relativity. The mathematical relationship between relativistic and oblateness effects will now be examined in detail.

Let r, θ be the polar coordinates introduced previously, and let t be the proper time and t_i the inertial time, that is, t is measured by a clock at the point (r, θ), and t_i is measured by an identical clock infinitely far removed from all matter. Then the relativistic formulation of the equations of motion of a particle of negligible mass about a body of mass M at the origin of coordinates consists of the following three equations

$$\left(1-\frac{2U}{c^2}\right)\left(\frac{dt_i}{dt}\right)^2 - \frac{1}{c^2-2U}\left(\frac{dr}{dt}\right)^2 - \frac{r^2}{c^2}\left(\frac{d\theta}{dt}\right)^2 = 1 \tag{78}$$

$$\frac{dt_i}{dt} = \frac{\sqrt{1+(2\mathbf{E}/c^2)}}{1-(2U/c^2)} \tag{79}$$

$$r^2 \frac{d\theta}{dt} = B \tag{80}$$

where c is the speed of light, U is the negative of the Newtonian gravitational potential of the mass M at the point (r, θ), $\mathbf{E}$ is the total energy per unit mass of the particle, and B is the angular momentum per unit mass of the particle. Equation (78) is Schwarzschild's solution of Einstein's field equations (ref. 14, p. 166, eq. 4.21a), equation (79) is the "clock-equation" (ref. 8, p. 182, eqs. 6, 7, 10), and equation (80) is the law of conservation of angular momentum.

These equations can be transformed into the notation of the present report by means of the following definitions of dimensionless variables:

$$\left.\begin{aligned} P_\varphi &= B/\sqrt{\mu R} \\ \xi &= (1/2)P_\varphi^2 \\ \eta &= R\mathbf{E}/\mu \\ \tau &= t\sqrt{\mu/R^3} \\ \tau_i &= t_i\sqrt{\mu/R^3} \\ u &= R/r, \qquad \nu = \mu/Rc^2 \end{aligned}\right\} \tag{81}$$

Eliminating t_i between equations (78) and (79) and transforming to the dimensionless variables gives

$$\xi\left(\frac{du}{d\theta}\right)^2 = \eta + \frac{RU}{\mu} - \xi u^2\left(1-\frac{2U}{c^2}\right) \tag{82}$$

$$\frac{d\tau}{d\theta} = \frac{1}{P_\varphi u^2} \tag{83}$$

$$\frac{d\tau_i}{d\theta} = \frac{\sqrt{1+2\nu\eta}}{P_\varphi u^2\left(1-\frac{2U}{c^2}\right)} \tag{84}$$

For the intermediate orbit about an oblate planet,

$$U = \frac{\mu}{R}\left(u + \frac{1}{3}\sigma u^3\right) \tag{85}$$

giving,

$$\xi\left(\frac{du}{d\theta}\right)^2 = \eta + \mu - \xi u^2 + \frac{1}{3}\sigma' u^3 + \frac{2}{3}\sigma\nu\xi u^5 \tag{86}$$

with

$$\sigma' = \sigma + 6\nu\xi \tag{87}$$

For the earth, $\nu = 6.95\times10^{-10}$ while $\sigma = 0(10^{-3})$. Thus, in equation (86), the ratio of the quintic to the cubic term is, for bounded orbits

$$2\nu\xi u^2 \approx \nu p u^2 < \frac{\nu(1+\epsilon)^2}{p} < 3\times10^{-9}$$

so that the quintic term can safely be neglected! Similarly, the second-order terms like $\sigma\nu$, $k^2\nu$,

ν^2 will be neglected throughout. The equations can then be reduced to the following form

$$\left.\begin{aligned} \xi\left(\frac{du}{d\theta}\right)^2 &= \eta+u-\xi u^2+\frac{1}{3}\sigma' u^3 \\ \frac{d\tau}{d\theta} &= \frac{1}{P_\varphi u^2} \\ \frac{d(\tau_i-\tau)}{d\theta} &= \nu\eta\frac{d\tau}{d\theta}+\frac{2\nu}{P_\varphi u} \end{aligned}\right\} \tag{88}$$

The solution of the first two of equations (88) has been obtained:

$$u=\frac{1}{a(1-\epsilon\cos E)}$$

$$n\tau=E-\epsilon\sin E+0(\sigma')$$

The third of equations (88) can be integrated by transforming to the eccentric anomaly, E, as independent variable, and neglecting second-order terms. Its solution is

$$n(\tau_i-\tau)=\frac{\nu}{2a}(3E+\epsilon\sin E)$$

with perigee at the time origin. When the periodic component is neglected the secular portion can be written in the form

$$t_i-t=\frac{3\nu}{2a}t=\frac{3\nu}{2a}t_i \tag{89}$$

To eliminate the inertial time, consider equation (78) applied to a clock on the earth, with its time denoted by t_e. With $dr/dt=0$ and $d\theta/dt=7\times10^{-5}$, it is found that

$$t_i=(1+\nu)t_e$$

and inserting this in equation (89) gives

$$t_e-t=\nu t_e\left(\frac{3}{2a}-1\right) \tag{90}$$

Thus, a clock in an orbit whose semimajor axis is 3/2 the radius of the earth would show no secular deviation from an identical clock on the earth. For smaller orbits, the satellite clock would run more slowly than the earth-bound clock, and for larger orbits the satellite clock would run faster. Inserting the numerical value of ν for the earth gives

$$(t_e-t)_{\text{seconds}}=0.0219\left(\frac{3}{2a}-1\right)(t_e)_{\text{years}}$$

Thus, the maximum attainable difference in clock rates is 0.02 second/year, which is probably not detectable by current techniques.

Similarly, the relativistic effect on the shape of the orbit is equally insignificant. Inserting equation (87) into equation (62) gives the relativistic effect on the advance of perigee as

$$\Delta\theta=\frac{1080\nu}{p}\text{ degrees/revolution}$$

In the worst case ($p=1$) this gives

$$\Delta\theta=0''.0027\text{/revolution}$$

and this quantity decreases linearly as the size of the orbit increases. In contrast, the advance of perigee due to oblateness is

$$\Delta\theta=\frac{360J}{p^2}\text{ degree/revolution}$$

In the worst case ($p=1$), this gives

$$\Delta\theta=0^\circ.584\text{/revolution}$$

DISCUSSION

The intermediate orbit presented in this report is essentially a closed-form solution of the main problem of artificial satellite theory; it is exact in the case of equatorial orbits, and, as indicated by the numerical example, is quite accurate for inclined orbits. (It can be called a closed-form solution because of the extremely rapid convergence of the series expansions.) As in the classical two-body problem, the kinematics of the orbit are described by Kepler's equation. When the time is prescribed, this is an implicit equation, containing as unknowns the two anomalies, E and v. Their values can be determined by an obvious iterative procedure involving both Kepler's equations and the classical relations between E and v, equations (54). (In the numerical example this was unnecessary because of the extremely small value of the modulus k, which permitted M_2 to be neglected, thus eliminating v from Kepler's equation.)

The qualitative nature of the approximation can be deduced heuristically from equations (11). The neglected portion of the Hamiltonian, H_1, is proportional to $\cos 2\varphi$, a strictly periodic function. This suggests that the intermediate orbit is exact in its treatment of secular phenomena, and is in error only in the periodic terms. This conjecture receives partial confirmation from the numerical example; the errors seem to oscillate, and even after 30 days no systematic growth is apparent.

Two further steps are needed to improve the theory. One is to solve the perturbation equations (appendix C) to determine the effects of the portion, H_1, of the Hamiltonian. The other is to include higher harmonics in the earth's gravitational potential. This second step could be split into two by separating the higher order terms into secular and harmonic portions, as was done in equations (11). These steps should provide significant improvements.

Ames Research Center
National Aeronautics and Space Administration
Moffet Field, Calif., *Feb. 2, 1961*

APPENDIX A

COORDINATE TRANSFORMATIONS

The transformation equations between astronomical (ρ, α, δ) and orbital coordinates (ρ, Ω, I, φ) are obtainable by either vector methods or the methods of spherical trigonometry (see ref. 5). The following redundant set is sufficient for the purposes of this report.

$$\left.\begin{aligned}
\sin\delta &= \sin I \sin\varphi \\
\cos\varphi &= \cos\delta\cos(\alpha-\Omega) \\
\cos I &= \cos\delta\sin\beta \\
\cos I\sin\varphi &= \cos\delta\sin(\alpha-\Omega) \\
\sin I\cos\varphi &= \cos\delta\cos\beta \\
\tan(\alpha-\Omega) &= \cos I\tan\varphi
\end{aligned}\right\} \tag{A1}$$

Differentiating gives the partial derivatives of the transformation from orbital to astronomical coordinates:

$$\left.\begin{aligned}
\frac{\partial\delta}{\partial\varphi} &= \frac{\sin I\cos\varphi}{\cos\delta} = \cos\beta \\
\frac{\partial\delta}{\partial I} &= \frac{\cos I\sin\varphi}{\cos\delta} = \sin\beta\sin\varphi \\
\frac{\partial\delta}{\partial\Omega} &= 0 \\
\frac{\partial\alpha}{\partial\varphi} &= \frac{\cos I}{\cos^2\delta} = \frac{\sin\beta}{\cos\delta} \\
\frac{\partial\alpha}{\partial I} &= -\frac{\sin\varphi\cos\varphi\sin I}{\cos^2\delta} = -\frac{\sin\varphi\cos\beta}{\cos\delta} \\
\frac{\partial\alpha}{\partial\Omega} &= 1
\end{aligned}\right\} \tag{A2}$$

Now if S is a function of position, so that

$$S = S(\rho, \alpha, \delta) = S(\rho, \Omega, I, \varphi) \tag{A3}$$

then the partial derivatives $\partial S/\partial\Omega$, $\partial S/\partial I$, $\partial S/\partial\varphi$ are not independent, but satisfy equation (4). To prove this, take the partial derivatives of equation (A3):

$$\frac{\partial S}{\partial\Omega} = \frac{\partial S}{\partial\alpha}\frac{\partial\alpha}{\partial\Omega} + \frac{\partial S}{\partial\delta}\frac{\partial\delta}{\partial\Omega}$$

$$\frac{\partial S}{\partial I} = \frac{\partial S}{\partial\alpha}\frac{\partial\alpha}{\partial I} + \frac{\partial S}{\partial\delta}\frac{\partial\delta}{\partial I}$$

$$\frac{\partial S}{\partial\varphi} = \frac{\partial S}{\partial\alpha}\frac{\partial\alpha}{\partial\varphi} + \frac{\partial S}{\partial\delta}\frac{\partial\delta}{\partial\varphi}$$

If this is regarded as a set of three simultaneous equations in two unknowns, $\partial S/\partial\alpha$ and $\partial S/\partial\delta$, then the determinant of the augmented matrix must vanish:

$$\begin{vmatrix}
\frac{\partial S}{\partial\Omega} & \frac{\partial\alpha}{\partial\Omega} & \frac{\partial\delta}{\partial\Omega} \\
\frac{\partial S}{\partial I} & \frac{\partial\alpha}{\partial I} & \frac{\partial\delta}{\partial I} \\
\frac{\partial S}{\partial\varphi} & \frac{\partial\alpha}{\partial\varphi} & \frac{\partial\delta}{\partial\varphi}
\end{vmatrix} = 0$$

Expanding and using equations (A2) gives equation (4)

$$\frac{\partial S}{\partial\Omega} = \frac{\partial S}{\partial\varphi}\cos I - \frac{\partial S}{\partial I}\sin I\operatorname{ctn}\varphi$$

Thus the formulation as a canonical, Hamiltonian system (eqs. (7) and (8)) is valid for any disturbance function, S, that depends only on position.

APPENDIX B

PHYSICAL SIGNIFICANCE OF THE CANONICAL CONSTANTS

The formation of the problem in terms of the Hamilton-Jacobi equation consist of equations (15) to (21). While this method of solution was not used, it is important to discuss it in order to prepare for future work involving the perturbation equations of appendix C. In particular, such work will require a physical interpretation of the canonical constants.

Equations (15), (18), and (30) give immediately

$$\eta = P_\rho = -\frac{\partial \mathbf{F}}{\partial \tau} = H_0 = \text{total energy}$$

Equations (16) and (18) give

$$P_\varphi = \frac{\partial \mathbf{F}}{\partial \varphi} = p_\varphi = \text{angular momentum}$$

$$P_\Omega = \frac{\partial \mathbf{F}}{\partial \Omega} = p_\Omega = \text{polar component of angular momentum}$$

Equations (5), (16), (18), and (21) give

$$\frac{d\rho}{d\tau} = \sqrt{2h\left(\frac{1}{\rho}\right)} = \frac{dF}{d\rho} \tag{21}$$

where h is the cubic

$$\left.\begin{aligned} h\left(\frac{1}{\rho}\right) &= P_\rho + \frac{1}{\rho} - \frac{1}{2}\frac{P_\varphi^2}{\rho^2} + \frac{\sigma}{3\rho^3} \\ \sigma &= J\left(\frac{3}{2}\frac{P_\Omega^2}{P_\varphi^2} - \frac{1}{2}\right) \end{aligned}\right\} \tag{B1}$$

Integrating equation (21) formally gives

$$F = \int_{\rho_0}^{\rho} \sqrt{2h}\, d\rho \tag{B2}$$

where ρ_0 is an arbitrary constant. Equations (17), (18), and (B2) give

$$Q_\rho = -\tau + \int_{\rho_0}^{\rho} \frac{1}{\sqrt{2h}}\, d\rho$$

$$Q_\varphi = \varphi + \int_{\rho_0}^{\rho} \frac{1}{\sqrt{2h}}\left(-\frac{P_\varphi}{\rho^2} - \frac{JP_\Omega^2}{\rho^3 P_\varphi^3}\right) d\rho$$

$$Q_\Omega = \Omega + \int_{\rho_0}^{\rho} \frac{1}{\sqrt{2h}}\, \frac{JP_\Omega}{\rho^3 P_\varphi^2}\, d\rho$$

Equations (21) and (12) then give

$$Q_\rho = -\tau + \int_{\rho_0}^{\rho} \frac{d\tau}{d\rho}\, d\rho$$

$$Q_\varphi = \varphi - \int_{\rho_0}^{\rho} \frac{d\tau}{d\rho}\, \frac{d\varphi}{d\tau}\, d\rho$$

$$Q_\Omega = \Omega - \int_{\rho_0}^{\rho} \frac{d\tau}{d\rho}\, \frac{d\Omega}{d\tau}\, d\rho$$

Thus, if ρ_0 is perigee,

$-Q_\rho = \tau_p$, time of perigee passage

$Q_\varphi = \varphi_p$, argument of perigee

$Q_\Omega = \Omega_p$, right ascension of the node at time of perigee passage

This completes the identification of the constants. They are the natural generalization of the canonical constants in the classical two-body problem (ref. 7, p. 148).

APPENDIX C

THE PERTURBATION EQUATIONS

GENERAL THEORY OF CONTACT TRANSFORMATIONS WITH DISSIPATIVE FORCES

The general theory of contact transformations for conservative systems is well known (see, e.g., ref. 7, ch. 10). The purpose of this section is to extend the theory to include dissipative forces. To do this, consider a canonical system

$$\left.\begin{aligned}\frac{dq_i}{d\tau}&=\frac{\partial H}{\partial p_i}+E_i\\ \frac{dp_i}{d\tau}&=-\frac{\partial H}{\partial q_i}+F_i\end{aligned}\right\} i=1,2,3,\ldots \tag{C1}$$

with generalized coordinates, q, momenta, p, and forces E, F. Let the Hamiltonian be split arbitrarily

$$H=H_0+H_1$$

and let

$$\left.\begin{aligned}q_i&=q_i(\tau,P,Q)\\ p_i&=p_i(\tau,P,Q)\end{aligned}\right\} \tag{C2}$$

be a solution of the reduced system

$$\left.\begin{aligned}\frac{\partial q_i}{\partial\tau}&=\frac{\partial H_0}{\partial p_i}\\ \frac{\partial p_i}{\partial\tau}&=-\frac{\partial H_0}{\partial q_i}\end{aligned}\right\} \tag{C3}$$

where P, Q are a shorthand notation for the six constants, P_1, P_2, P_3, Q_1, Q_2, Q_3. The equations (C2) can be made to satisfy the complete system (C1) by allowing the constants P, Q to become variables. This transforms the system (C1) into the system

$$\left.\begin{aligned}\frac{\partial q_i}{\partial P_j}\frac{dP_j}{d\tau}+\frac{\partial q_i}{\partial Q_j}\frac{dQ_j}{d\tau}&=\frac{\partial H_1}{\partial p_i}+E_i\\ \frac{\partial p_i}{\partial P_j}\frac{dP_j}{d\tau}+\frac{\partial p_i}{\partial Q_j}\frac{dQ_j}{d\tau}&=-\frac{\partial H_1}{\partial q_i}+F_i\end{aligned}\right\} \tag{C4}$$

where use has been made of Einstein's summation convention that a repeated subscript is to be summed over the values 1, 2, 3.

Multiplying the first of equations (C4) by $\partial p_i/\partial Q_k$, the second by $-\partial q_i/\partial Q_k$ and adding gives

$$[P_j,Q_k]\frac{dP_j}{d\tau}+[Q_j,Q_k]\frac{dQ_j}{d\tau}=\frac{\partial H_1}{\partial Q_k}+X_k$$

where

$$X_k=E_i\frac{\partial p_i}{\partial Q_k}-F_i\frac{\partial q_i}{\partial Q_k} \tag{C5}$$

and the Lagrangian brackets are defined by

$$[x,y]=-[y,x]=\frac{\partial q_i}{\partial x}\frac{\partial p_i}{\partial y}-\frac{\partial q_i}{\partial y}\frac{\partial p_i}{\partial x}$$

x, y denoting any of the variables P, Q.

Similarly, multiplying the first of equations (C4) by $\partial p_i/\partial P_k$, the second by $-\partial q_i/\partial P_k$, and adding gives

$$[P_j,P_k]\frac{dP_j}{d\tau}+[Q_j,P_k]\frac{dQ_j}{d\tau}=\frac{\partial H_1}{\partial P_k}+Y_k$$

where

$$Y_k=E_i\frac{\partial p_i}{\partial P_k}-F_i\frac{\partial q_i}{\partial P_k} \tag{C6}$$

Now, if the transformation from (p, q) to (P, Q) is a contact transformation, the Lagrangian brackets reduce to (ref. 7, ch. 10)

$$[Q_j,Q_k]=[P_j,P_k]=0$$

$$[Q_j,P_k]=\begin{cases}0 \text{ if } j\neq k\\ 1 \text{ if } j=k\end{cases}$$

and the variables P, Q satisfy the canonical system

$$\left.\begin{aligned}\frac{dQ_k}{d\tau}&=\frac{\partial H_1}{\partial P_k}+Y_k\\ \frac{dP_k}{d\tau}&=-\frac{\partial H_1}{\partial Q_k}-X_k\end{aligned}\right\} \tag{C7}$$

This is the general formulation of the theory of contact transformations with dissipative forces.

PERTURBATION EQUATIONS FOR ARTIFICIAL SATELLITE THEORY

To apply the results of the preceding section to artificial satellite theory, identify the subscripts 1, 2, 3 with ρ, φ, Ω, respectively. Instead of using the P, Q notation, it is convenient to take as new coordinates (see appendix B)

$$\left.\begin{aligned}\varphi_p&=Q_\varphi\\ \Omega_p&=Q_\Omega\\ \eta&=P_\rho\end{aligned}\right\} \tag{C8}$$

and, as their conjugate momenta,

$$\left.\begin{aligned}P_\varphi&=p_\varphi\\ P_\Omega&=p_\Omega\\ \tau_p&=-Q_\rho\end{aligned}\right\} \tag{C9}$$

The generalized forces E, F are found by identifying equation (C1) with equation (8):

$$\left.\begin{aligned}E_\rho&=0\\ E_\varphi&=-\frac{\rho P_\Omega D_I\sin\varphi}{P_\varphi^2\sin I}\\ E_\Omega&=\frac{\rho D_I\sin\varphi}{P_\varphi\sin I}\\ F_\rho&=D_\rho\\ F_\varphi&=\rho D_\varphi\\ F_\Omega&=\rho(D_\varphi\cos I-D_I\sin I\cos\varphi)\end{aligned}\right\} \tag{C10}$$

with, of course,

$$I=\text{arc}\cos P_\Omega/P_\varphi$$

Since φ_p occurs only in the expression for φ and Ω_p only in the expression for Ω (eqs. (64)), it follows that

$$\frac{\partial\varphi}{\partial\varphi_p}=1 \qquad \frac{\partial\Omega}{\partial\Omega_p}=1$$

and all the other partial derivatives with respect to φ_p and Ω_p vanish. Similarly, from equations (C9)

$$\frac{\partial p_\varphi}{\partial P_\varphi}=1 \qquad \frac{\partial p_\Omega}{\partial P_\Omega}=1$$

and all the other partial derivatives of p_φ and p_Ω vanish. Also, from equation (11),

$$H_1=\frac{J}{2\rho^3}\left(\frac{P_\Omega^2}{P_\varphi^2}-1\right)\cos 2\varphi \tag{C11}$$

so that H_1 does not contain Ω and contains φ_p only via φ. Hence

$$\frac{\partial H_1}{\partial\Omega}=0 \qquad \frac{\partial H_1}{\partial\varphi_p}=\frac{\partial H_1}{\partial\varphi}$$

Combining these results with equations (C5) to (C7) gives the canonical system of perturbation equations for artificial satellite theory:

$$\left.\begin{aligned}\frac{\partial\varphi_p}{d\tau}&=E_\varphi+\frac{\partial H_1}{\partial P_\varphi}-F_\rho\frac{\partial\rho}{\partial P_\varphi}-F_\varphi\frac{\partial\varphi}{\partial P_\varphi}-F_\Omega\frac{\partial\Omega}{\partial P_\varphi}\\ \frac{d\Omega_p}{d\tau}&=E_\Omega+\frac{\partial H_1}{\partial P_\Omega}-F_\rho\frac{\partial\rho}{\partial P_\Omega}-F_\varphi\frac{\partial\varphi}{\partial P_\Omega}-F_\Omega\frac{\partial\Omega}{\partial P_\Omega}\\ \frac{d\eta}{d\tau}&=\frac{\partial H_1}{\partial\tau_p}-F_\rho\frac{\partial\rho}{\partial\tau_p}-F_\varphi\frac{\partial\varphi}{\partial\tau_p}-F_\Omega\frac{\partial\Omega}{\partial\tau_p}\end{aligned}\right\} \tag{C12}$$

$$\left.\begin{aligned}\frac{dP_\varphi}{d\tau}&=-\frac{\partial H_1}{\partial\varphi}+F_\varphi\\ \frac{dP_\Omega}{d\tau}&=F_\Omega\\ \frac{d\tau_p}{d\tau}&=-\frac{\partial H_1}{\partial\eta}+F_\rho\frac{\partial\rho}{\partial\eta}+F_\varphi\frac{\partial\varphi}{\partial\eta}+F_\Omega\frac{\partial\Omega}{\partial\eta}\end{aligned}\right\} \tag{C13}$$

Equation (C11) could be generalized to include higher harmonics in the earth's gravitational potential without invalidating any of the subsequent analysis.

Equations (C12) and (C13), then, are the perturbation equations that must be solved to improve the intermediate orbit. While no attempt will be made to solve them here, it may be remarked that they are subject to the usual difficulty: the right members contain the old coordinates (ρ, φ, Ω) as well as the new coordinates and momenta. Furthermore, the old coordinates have not been expressed explicitly in terms of τ

and the new variables. Rather, the coordinates have been expressed in terms of the new variables, the true anomaly, v, and intermediate parameters like a, p, ϵ, k, f, etc.

The true anomaly is related to τ by an implicit equation, namely, Kepler's equation. It is for this reason that so much emphasis has been placed on series expansions (see appendix E), since they form the only basis for attacking the problem of expressing the partial derivatives in equations (C12) and (C13) in terms of the new variables.

APPENDIX D

CATALOG OF ORBITS

While the main purpose of this report is to develop the theory of satellite orbits, other types of orbits merit some mention. In this appendix the complete catalog of types will be given, without proofs. In every case the proof consists merely in solving the characteristic equation (eq. (33)) and then integrating the energy equation (eq. (34)). When elliptic functions and integrals occur, reference will be made to the appropriate formula of reference 11.

TYPE 1. $0<\xi^2<\sigma$

The equation of the orbit is (ref. 11, eq. 243.00)

$$u=u_1+A\tan^2\psi$$

where

$$u_1=\frac{1}{\sigma}(\xi-2\sqrt{\sigma-\xi^2}\sinh\zeta)$$

$$A=\frac{1}{\sigma}\sqrt{3(\sigma-\xi^2)(1+4\sinh^2\zeta)}$$

and the angles ζ and ψ are defined by

$$\sinh 3\zeta=\frac{3\sigma^2\eta+3\sigma\xi-2\xi^3}{2(\sigma-\xi^2)^{3/2}}$$

$$\sin 2\psi=sn\ 2f\theta$$

$$\theta=\frac{1}{2f}F(2\psi,k)$$

$$f=\frac{1}{2}\sqrt{\frac{\sigma A}{3\xi}}$$

$$k^2=\frac{1}{2}+\frac{1}{2}\sinh\zeta\sqrt{\frac{3}{1+4\sinh^2\zeta}}$$

If $\eta<\eta_T$, then $u>u_1>1$, $\rho<1$, and the orbit is entirely inside the planet. If $\eta>\eta_T$, then $u_1<1$, and the orbit intersects the surface of the planet at

$$\psi=\text{arc tan}\sqrt{\frac{1-u_1}{A}}$$

The orbit has no perigee; instead it approaches the center of the planet ($\rho=0$, $u=\infty$) as

$$\psi\rightarrow\frac{\pi}{2}$$

$$\theta\rightarrow\frac{1}{f}K(k)$$

If $\eta<0$, the orbit has an apogee at

$$\theta=0 \qquad u=u_1$$

If $\eta\geq 0$, the orbit is a captive escape orbit, with the asymptotic direction (as $\rho\rightarrow\infty$)

$$\psi=\text{arc tan}\sqrt{\frac{-u_1}{A}}$$

For realizable orbits, $\eta>\eta_T$ and

$$\frac{1}{2}-\frac{1}{4}\sqrt{\frac{3\sigma}{1+\sigma}}<k^2<\frac{1}{2}+\frac{1}{4}\sqrt{3}$$

Thus, for equatorial orbits about the earth, with $\sigma=0.0016$,

$$0.4827<k^2<0.9330$$

For this range of values of k, the transformation equations between θ and ψ can be approximated by

$$\sin 2\psi=\frac{1}{\sqrt{k}}\tanh\frac{\pi f\theta}{K'}$$

$$\theta=\frac{K'}{2\pi f}\ ln\frac{1+\sqrt{k}\sin 2\psi}{1-\sqrt{k}\sin 2\psi}$$

where K' is the complete elliptic integral of the first kind with the complementary modulus

$$k'=\sqrt{1-k^2}$$

(see ref. 11, eq. 127.02).

TYPE 2. $\xi^2=\sigma>0$

This is simply a special case of type 1, with $\zeta \to \infty$:

$$A=\sqrt{\frac{3}{\sigma}}\sqrt[3]{|1+3\eta\sqrt{\sigma}|}$$

$$k^2=\begin{cases}\frac{1}{2}+\frac{1}{4}\sqrt{3} & \text{if} \quad \eta>\frac{-1}{3\sqrt{\sigma}}\\ \frac{1}{2}-\frac{1}{4}\sqrt{3} & \text{if} \quad \eta<\frac{-1}{3\sqrt{\sigma}}\end{cases}$$

There is also the degenerate case when $\eta=-1/(3\sqrt{\sigma})$. The characteristic equation has a triple root, and the equation of the orbit is

$$u=\frac{1}{\sqrt{\sigma}}\left(1+\frac{12}{\theta^2}\right)$$

Since $\sigma<1$, $u>1$, $\rho<1$, and this orbit lies entirely inside the planet.

TYPE 3. $\xi^2>\sigma$, $\eta>\eta_1$

The orbit equation is formally the same as for type 1:

$$u=u_1+A\tan^2\psi$$

but now the parameters are defined by the following equations:

$$u_1=\frac{\xi-2\sqrt{\xi^2-\sigma}\cosh\zeta}{\sigma}$$

$$A=\frac{1}{\sigma}\sqrt{3(\xi^2-\sigma)(4\cosh^2\zeta-1)}$$

$$\cosh^2\frac{3\zeta}{2}=\frac{\eta-\eta_2}{\eta_1-\eta_2},\ \zeta>0$$

$$\cosh 3\zeta=\frac{3\sigma^2\eta+3\sigma\xi-2\xi^3}{2(\xi^2-\sigma)^{3/2}}$$

$$\sin 2\psi=sn 2f\theta$$

$$\theta=\frac{1}{2f}F(2\psi, k)$$

$$f=\frac{1}{2}\sqrt{\frac{\sigma A}{3\zeta}}$$

$$k^2=\frac{1}{2}+\frac{1}{2}\cosh\zeta\sqrt{\frac{3}{4\cosh^2\zeta-1}}$$

Thus

$$\frac{1}{2}+\frac{1}{4}\sqrt{3}<k^2<1$$

and the same approximations can be used as for type 1.

If $\sigma>0$, there is no essential difference from type 1. But if $\sigma<0$, there is an internal perigee at

$$\rho=\frac{1}{u_1}\leq\sqrt{\frac{-\sigma}{3}}$$

and, at perigee, the orbit is convex toward the center of the planet. Outside the planet, the orbit is essentially like those of type 1.

TYPE 4. $\xi^2>\sigma$, $\eta=\eta_1$

The energy equation has two solutions:

$$u=u_1+A\operatorname{ctnh}^2 f\theta$$

$$u=u_1+A\tanh^2 f\theta$$

where

$$u_1=\frac{\xi-2\sqrt{\xi^2-\sigma}}{\sigma}$$

$$A=\frac{3}{\sigma}\sqrt{\xi^2-\sigma}$$

The first solution is unrealizable; if $\sigma<0$, then $u<0$, $\rho<0$, which is meaningless; if $\sigma>0$, then $u>1$, $\rho<1$, and the orbit lies entirely inside the planet.

The second solution is simply a special case of that of type 3, with $\xi=0$. But now, if $\sigma>0$, there is no perigee. Instead, the internal portion of the orbit is a spiral that approaches the circle $\rho=\sqrt{\sigma}$ as $\theta\to\infty$. If $\sigma<0$, the orbit is essentially like that of type 3.

TYPE 5. $\xi^2>\sigma$, $\eta<\eta_2$

This type is obtainable formally from type 3 simply by changing the sign of $\cosh\zeta$. But now the orbit is imaginary if $\sigma<0$ and internal if $\sigma>0$.

TYPE 6. $\xi^2>\sigma$, $\eta=\eta_2$

There are two solutions. One is the degenerate case of type 5 with $\zeta=\infty$. The other solution is

$$u=\frac{\xi-\sqrt{\xi^2-\sigma}}{\sigma}$$

that is, a circular orbit. If $\xi<(1+\sigma)/2$, the orbit lies inside the planet, otherwise outside.

TYPE 7. $\xi^2>\sigma$, $\eta_2<\eta<\eta_1$

This type is, of course, the one discussed in the body of the report. For the captive orbits, the eccentricity ϵ, and modulus k, can both be large, so that the series expansions may not be valid. In this case, other expansions of the elliptic functions must be used (see ref. 11, eqs. 125.02, 126.01).

APPENDIX E

SERIES EXPANSIONS

The series expansions given in the section on satellite orbits are based on the definitions of η_2 and ϵ_0, equations (37) and (56). Expanding the first by the binomial theorem and inserting the result in the second gives

$$\epsilon_0^2=4\xi\eta+\sum_{m=2}^{\infty}\frac{8}{3}\binom{3/2}{m}(-\lambda)^m=1+4\xi\eta+\frac{1}{6}\lambda+\frac{1}{16}\lambda^2+\frac{1}{32}\lambda^3+\frac{7}{384}\lambda^4+\frac{3}{256}\lambda^5\ldots \tag{E1}$$

Equation (40) can be transformed, by means of equations (37) and (56), into

$$\zeta=\frac{2}{3}\text{ arc sin}\left[\frac{\epsilon_0|\lambda|\sqrt{3}}{4}(1-\lambda)^{-3/4}\right]$$

The binomial theorem and the Maclaurin's series for the inverse sine give

$$\zeta=\frac{1}{6}\epsilon_0|\lambda|\sqrt{3}\left[1+\frac{3}{4}\lambda+\frac{21+\epsilon_0^2}{32}\lambda^2+\frac{77+9\epsilon_0^2}{128}\lambda^3+\left(\frac{1155}{2048}+\frac{117\epsilon_0^2}{1024}+\frac{27\epsilon_0^4}{10240}\right)\lambda^4+\left(\frac{4389}{8192}+\frac{663\epsilon_0^2}{4096}+\frac{81\epsilon_0^4}{8192}\right)\lambda^5\ldots\right]$$

From this the successive powers of ζ can be obtained and thence the trigonometric functions. The useful ones are

$$\sin\zeta=\frac{1}{6}\epsilon_0|\lambda|\sqrt{3}\left[1+\frac{3}{4}\lambda+\frac{189+5\epsilon_0^2}{288}\lambda^2+\frac{77+5\epsilon_0^2}{128}\lambda^3+\left(\frac{1155}{2048}+\frac{65\epsilon_0^2}{1024}+\frac{77\epsilon_0^4}{55296}\right)\lambda^4+\left(\frac{4389}{8192}+\frac{1105\epsilon_0^2}{12288}+\frac{385\epsilon_0^4}{73728}\right)\lambda^5\ldots\right] \tag{E2}$$

$$\cos\zeta=1-\frac{\epsilon_0^2}{24}\lambda^2-\frac{\epsilon_0^2}{16}\lambda^3-\left(\frac{5\epsilon_0^2}{64}+\frac{\epsilon_0^4}{432}\right)\lambda^4-\left(\frac{35\epsilon_0^2}{384}+\frac{\epsilon_0^4}{144}\right)\lambda^5-\left(\frac{105\epsilon_0^2}{1024}+\frac{\epsilon_0^4}{72}+\frac{7\epsilon_0^6}{31104}\right)\lambda^6\ldots \tag{E3}$$

and, if $\sigma>0$,

$$\sin\left(\zeta+\frac{\pi}{3}\right)=\frac{1}{2}\sqrt{3}\left[1+\frac{\epsilon_0}{6}\lambda+\left(\frac{\epsilon_0}{8}-\frac{\epsilon_0^2}{24}\right)\lambda^2+\left(\frac{7\epsilon_0}{64}-\frac{\epsilon_0^2}{16}+\frac{5\epsilon_0^3}{1728}\right)\lambda^3+\left(\frac{77\epsilon_0}{768}-\frac{5\epsilon_0^2}{64}+\frac{5\epsilon_0^3}{768}-\frac{\epsilon_0^4}{432}\right)\lambda^4+\left(\frac{385\epsilon_0}{4096}-\frac{35\epsilon_0^2}{384}+\frac{65\epsilon_0^3}{6144}-\frac{\epsilon_0^4}{144}+\frac{77\epsilon_0^5}{331776}\right)\lambda^5\ldots\right] \tag{E4}$$

while the sign of ϵ_0 must be reversed when $\sigma<0$.

It is now simply a matter of algebra to substitute these series into the definitions of the parameters, k, f, p, etc., giving the following expansions:

$$p=2\xi\left(1-\frac{1+\xi\eta}{3}\lambda-\frac{1+\xi\eta-\xi^2\eta^2}{9}\lambda^2-\frac{2+3\xi\eta-\xi^2\eta^2+\xi^3\eta^3}{27}\lambda^3-\frac{5+10\xi\eta+\xi^3\eta^3-\xi^4\eta^4}{81}\lambda^4\ldots\right) \tag{E5}$$

$$\epsilon=\epsilon_0\left[1-\frac{1+4\xi\eta}{12}\lambda-\left(\frac{1}{48}+\frac{\xi\eta}{8}-\frac{\xi^2\eta^2}{9}\right)\lambda^2-\left(\frac{1}{108}+\frac{85\xi\eta}{864}-\frac{\xi^2\eta^2}{24}+\frac{\xi^3\eta^3}{27}\right)\lambda^3-\left(\frac{107}{20736}+\frac{235\xi\eta}{2592}-\frac{\xi^2\eta^2}{384}+\frac{\xi^3\eta^3}{72}-\frac{\xi^4\eta^4}{81}\right)\lambda^4\ldots\right] \tag{E6}$$

Squaring this and using (E1) gives

$$\epsilon^2=1+4\xi\eta-4\xi\eta\,\frac{1+2\xi\eta}{3}\,\lambda-4\xi\eta\,\frac{1+\xi\eta-3\xi^2\eta^2}{9}\,\lambda^2-4\xi\eta\,\frac{2+3\xi\eta-3\xi^2\eta^2+4\xi^3\eta^3}{27}\,\lambda^3$$
$$-4\xi\eta\,\frac{5+10\xi\eta-4\xi^2\eta^2+5\xi^3\eta^3-5\xi^4\eta^4}{81}\,\lambda^4\cdots \quad \text{(E7)}$$

The second of equations (55) can be solved for a in terms of η, σ, and p:

$$a=-\frac{1}{4\eta}-\frac{1}{4\eta}\sqrt{1+\frac{8\sigma\eta}{3p}}$$

The binomial theorem and equation (E5) then give

$$a=\frac{-1}{2\eta}\left(1+\frac{\xi\eta}{3}\lambda+\frac{\xi\eta}{9}\lambda^2+\frac{2\xi\eta+\xi^2\eta^2}{27}\lambda^3+\frac{5\xi\eta+5\xi^2\eta^2}{81}\lambda^4\cdots\right) \quad \text{(E8)}$$

Equations (47), (E2), and (E4) give, for $\sigma>0$,

$$k^2=\frac{1}{3}\epsilon_0\lambda\left[1+\left(\frac{3}{4}-\frac{\epsilon_0}{6}\right)\lambda+\left(\frac{21}{32}-\frac{\epsilon_0}{4}+\frac{25\epsilon_0^2}{288}\right)\lambda^2+\left(\frac{77}{128}-\frac{5\epsilon_0}{16}+\frac{25\epsilon_0^2}{128}-\frac{7\epsilon_0^3}{288}\right)\lambda^3\right.$$
$$\left.+\left(\frac{1155}{2048}-\frac{35\epsilon_0}{96}+\frac{325\epsilon_0^2}{1024}-\frac{7\epsilon_0^3}{96}+\frac{1967\epsilon_0^4}{165888}\right)\lambda^4\cdots\right] \quad \text{(E9)}$$

and again the sign of ϵ_0 must be reversed when $\sigma<0$.

Equations (47) and (E4) give, for $\sigma>0$,

$$f=\frac{1}{2}\left[1-\left(\frac{1}{4}-\frac{\epsilon_0}{12}\right)\lambda-\left(\frac{3}{32}-\frac{\epsilon_0}{24}+\frac{7\epsilon_0^2}{288}\right)\lambda^2-\left(\frac{7}{128}-\frac{\epsilon_0}{32}+\frac{35\epsilon_0^2}{1152}\right.\right.$$
$$\left.\left.-\frac{\epsilon_0^3}{288}\right)\lambda^3-\left(\frac{77}{2048}-\frac{5\epsilon_0}{192}+\frac{35\epsilon_0^2}{1024}-\frac{\epsilon_0^3}{144}+\frac{289\epsilon_0^4}{165888}\right)\lambda^4\cdots\right] \quad \text{(E10)}$$

and the sign of ϵ_0 must be reversed when $\sigma<0$.

The complete elliptic integral, $K(k)$, has a well-known expansion in powers of k (ref. 11, eq. 900.00):

$$K=\frac{\pi}{2}\left(1+\frac{k^2}{4}+\frac{9k^4}{64}+\frac{25k^6}{256}+\frac{1225k^8}{16384}+\frac{3969k^{10}}{65536}\cdots\right)$$

Inserting the series for k^2 (eq. (E9)) gives, for $\sigma>0$,

$$K=\frac{\pi}{2}\left[1+\frac{\epsilon_0}{12}\lambda+\left(\frac{\epsilon_0}{16}+\frac{\epsilon_0^2}{576}\right)\lambda^2+\left(\frac{7\epsilon_0}{128}+\frac{\epsilon_0^2}{384}+\frac{13\epsilon_0^2}{2304}\right)\lambda^3+\left(\frac{77\epsilon_0}{1536}+\frac{5\epsilon_0^2}{1536}+\frac{13\epsilon_0^3}{1024}+\frac{313\epsilon_0^4}{1327104}\right)\lambda^4\cdots\right] \quad \text{(E11)}$$

and the sign of ϵ_0 must be reversed when $\sigma<0$.

Combining with the series for f (eq. (E10)), gives the secular coefficients of equations (61):

$$\frac{K}{\pi f}=1+\frac{1}{4}\lambda+\left(\frac{35}{192}+\frac{5\xi\eta}{48}\right)\lambda^2+\left(\frac{385}{2304}+\frac{35\xi\eta}{192}\right)\lambda^3+\left(\frac{25025}{147456}+\frac{5005\xi\eta}{18432}+\frac{385\xi^2\eta^2}{9216}\right)\lambda^4\cdots \quad \text{(E12)}$$

$$\frac{\pi f}{K}=1-\frac{1}{4}\lambda-\left(\frac{23}{192}+\frac{5\xi\eta}{48}\right)\lambda^2-\left(\frac{211}{2304}+\frac{25\xi\eta}{192}\right)\lambda^3-\left(\frac{12269}{147456}+\frac{995\xi\eta}{6144}+\frac{95\xi^2\eta^2}{3072}\right)\lambda^4\cdots \quad \text{(E13)}$$

Note that when the series for K and f are combined, the odd powers of ϵ_0 disappear; eliminating the even powers by means of equation (E1) gives equations (E12) and (E13), which are independent of the sign of σ.

Jacobi's nome, q, has the well-known series representation (ref. 11, eq. 901.00)

$$q=\frac{k^2}{16}\left(1+\frac{k^2}{8}+\frac{15k^4}{256}+\frac{75k^6}{2048}+\frac{1707k^8}{65536}\ldots\right)^4$$

Inserting the series for k^2, equation (E9) gives, for $\sigma>0$,

$$q=\frac{\epsilon_0}{48}\lambda+\frac{\epsilon_0}{64}\lambda^2+\left(\frac{7\epsilon_0}{512}+\frac{13\epsilon_0^3}{9216}\right)\lambda^3+\left(\frac{77\epsilon_0}{6144}+\frac{13\epsilon_0^3}{4096}\right)\lambda^4\ldots \tag{E14}$$

and the sign of ϵ_0 must be reversed when $\sigma<0$.

Two other quantities that occur in certain Fourier series are

$$\frac{16q}{k^2}=1+\frac{\epsilon_0}{6}\lambda+\left(\frac{\epsilon_0}{8}+\frac{5\epsilon_0^2}{576}\right)\lambda^2+\left(\frac{7\epsilon_0}{64}+\frac{5\epsilon_0^2}{384}+\frac{13\epsilon_0^3}{1152}\right)\lambda^3+\left(\frac{77\epsilon_0}{768}+\frac{25\epsilon_0^2}{1536}+\frac{13\epsilon_0^3}{512}+\frac{3089\epsilon_0^4}{2654208}\right)\lambda^4\ldots \tag{E15}$$

and

$$2\,\frac{K-E}{Kk^2}=1+\frac{\epsilon_0}{24}\lambda+\frac{\epsilon_0}{32}\lambda^2+\epsilon_0\,\frac{833+308\xi\eta}{27648}\lambda^3+\epsilon_0\,\frac{10549+8316\xi\eta}{331776}\lambda^4\ldots \tag{E16}$$

Equations (E15) and (E16) are valid as written when $\sigma>0$; when $\sigma<0$, the sign of ϵ_0 must be reversed.

The series relating θ and v (eqs. (59)) are obtained as follows. Begin with the Fourier series expansions (ref. 11, eq. 806.01)

$$\left.\begin{aligned}\frac{1}{\sqrt{1-k^2\cos^2(v/2)}}&=\sum_{m=0}^{\infty}A_m\cos mv\\ \frac{1}{\sqrt{1-k^2\sin^2(v/2)}}&=\sum_{m=0}^{\infty}(-1)^mA_m\cos mv\end{aligned}\right\} \tag{E17}$$

where

$$\left.\begin{aligned}A_0&=\frac{2K}{\pi}\\ A_m&=2\sum_{j=m}^{\infty}\binom{2j}{j}\binom{2j}{j-m}\left(\frac{k}{4}\right)^{2j},\quad m=1,2,3,\ldots\end{aligned}\right\} \tag{E18}$$

The first four coefficients are

$$A_1=\frac{k^2}{4}+\frac{3k^4}{16}+\frac{75k^6}{512}+\frac{245k^8}{2048}\ldots$$

$$A_2=\frac{3k^4}{64}+\frac{15k^6}{256}+\frac{245k^8}{4096}\ldots$$

$$A_3=\frac{5k^6}{512}+\frac{35k^8}{2048}\ldots$$

$$A_4=\frac{35k^8}{16384}\ldots$$

Next define coefficients B_m by

$$B_m=\frac{A_m}{2f}\qquad\text{if}\quad\sigma>0$$

$$B_m=\frac{(-1)^mA_m}{2f}\qquad\text{if}\quad\sigma<0$$

If the series for k^2 and f are introduced, the usual reversal of sign of ϵ_0 is nullified by the factor $(-1)^m$, and the resulting expressions are independent of the sign of σ. The first four coefficients are

$$\left.\begin{aligned}B_1&=\frac{\epsilon_0\lambda}{12}\left[1+\lambda+\left(\frac{1259}{1152}+\frac{107\xi\eta}{288}\right)\lambda^2+\left(\frac{539}{432}+\frac{535\xi\eta}{576}\right)\lambda^3\ldots\right]\\ B_2&=\frac{\lambda^2}{192}\left[1+4\xi\eta+\left(\frac{23}{12}+7\xi\eta\right)\lambda+\left(\frac{5045}{1728}+\frac{1177\xi\eta}{108}+\frac{275\xi^2\eta^2}{108}\right)\lambda^2\ldots\right]\\ B_3&=\frac{5\epsilon_0\lambda^3}{13824}\left[1+4\xi\eta+\left(\frac{8}{3}+10\xi\eta\right)\lambda\ldots\right]\\ B_4&=\frac{35\,(1+8\xi\eta+16\xi^2\eta^2)}{1327104}\lambda^4\ldots\end{aligned}\right\} \tag{E19}$$

Inserting these Fourier series in equation (58) gives

$$\frac{d\theta}{dv}=\frac{K}{\pi f}+\sum_{m=1}^{\infty} B_m \cos mv \tag{E20}$$

Integrating term-by-term gives the first of equations (59)

$$\theta=\frac{K}{\pi f}\, v+\sum_{m=1}^{\infty}\frac{B_m}{m}\sin mv$$

Conversely, to express v as a function of θ, begin with the Fourier series (ref. 11, eqs. 908.08, 908.03)

$$\left.\begin{aligned} 2k'\, nd\, f\theta&=\frac{\pi}{K}+\frac{4\pi}{K}\sum_{m=1}^{\infty}\frac{(-q)^m}{1+q^{2m}}\cos\frac{m\pi f}{K}\,\theta, & \sigma>0\\ 2\, dn\, f\theta&=\frac{\pi}{K}+\frac{4\pi}{K}\sum_{m=1}^{\infty}\frac{q^m}{1+q^{2m}}\cos\frac{m\pi f}{K}\,\theta, & \sigma<0 \end{aligned}\right\} \tag{E21}$$

Inserting these series in equations (58) and integrating term-by-term gives the second of equations (59):

$$v=\frac{\pi f}{K}\,\theta-\sum_{m=1}^{\infty}\theta_m \sin\frac{m\pi f}{K}\,\theta$$

where

$$\left.\begin{aligned} \theta_m&=-\frac{4(-q)^m}{(1+q^{2m})\,m}\text{ if }\sigma>0\\ \theta_m&=-\frac{4q^m}{(1+q^{2m})\,m}\text{ if }\sigma<0 \end{aligned}\right\} \tag{E22}$$

By equation (E14), q contains only odd powers of ϵ_0, so that again the usual sign reversal is nullified by the factor $(-1)^m$, and the final expressions for θ_m are independent of the sign of σ. The first four coefficients are:

$$\left.\begin{aligned} \theta_1&=\frac{\epsilon_0\lambda}{16}\left[\frac{4}{3}+\lambda+\left(\frac{1667}{1728}+\frac{155\xi\eta}{432}\right)\lambda^2+\left(\frac{21127}{20736}+\frac{155\xi\eta}{192}\right)\lambda^3\ \ldots\right]\\ \theta_2&=\frac{-\lambda^2}{2304}\left[2+8\xi\eta+\left(\frac{10}{3}+12\xi\eta\right)\lambda+\left(\frac{55}{12}+\frac{103\xi\eta}{6}+\frac{13\xi^2\eta^2}{3}\right)\lambda^2\ \ldots\right]\\ \theta_3&=\frac{\epsilon_0\lambda^3}{82944}\left[1+4\xi\eta+\left(\frac{29}{12}+9\xi\eta\right)\lambda\ \ldots\right]\\ \theta_4&=-\frac{1+8\xi\eta+16\xi^2\eta^2}{5308416}\,\lambda^4\ \ldots \end{aligned}\right\} \tag{E23}$$

The inversion of equation (65) to obtain v explicitly as a function of φ is accomplished as follows. Assume an expansion of the form

$$v=\vartheta+\sum_{m=1}^{\infty} v_m \sin m\vartheta \tag{E24}$$

where

$$\vartheta=\frac{\varphi-\varphi_p}{\varphi_s}$$

Then, by the usual formula of Fourier analysis,

$$v_m=\frac{1}{\pi}\int_{-\pi}^{\pi}(v-\vartheta)\sin m\vartheta d\vartheta$$

Integrating by parts gives

$$v_m=\frac{1}{m\pi}\int_{-\pi}^{\pi}\cos m\vartheta dv$$

Hence, from equation (65)

$$v_m=\frac{1}{m\pi}\int_{-\pi}^{\pi}\cos m\left(v+\sum_{j=1}^{\infty}\frac{\varphi_j}{\varphi_s}\sin jv\right)dv$$

Expanding the cos into a Taylor's series centered at mv, and carrying out the necessary multiplications of Fourier series, gives, to the fourth order in J,

$$\left.\begin{aligned} v_1&=-\vartheta_1-\frac{1}{2}\vartheta_1\vartheta_2+\frac{1}{8}\vartheta_1^3\ \ldots\\ v_2&=-\vartheta_2+\frac{1}{2}\vartheta_1^2-\vartheta_1\vartheta_3+\vartheta_1^2\vartheta_2-\frac{1}{6}\vartheta_1^4\ \ldots\\ v_3&=-\vartheta_3+\frac{3}{2}\vartheta_1\vartheta_2-\frac{3}{8}\vartheta_1^3\ \ldots\\ v_4&=-\vartheta_4+\vartheta_2^2+2\vartheta_1\vartheta_3-2\vartheta_1^2\vartheta_2+\frac{1}{3}\vartheta_1^4\ \ldots \end{aligned}\right\} \tag{E25}$$

where

$$\vartheta_m=\varphi_m/\varphi_s$$

Since the equation of the orbit involves $\cos v$, it is desirable to have a series expansion for this quantity. From equation (E24),

$$\cos v = \cos \vartheta \cos\left(\sum_{m=1}^{\infty} v_m \sin m\vartheta\right) - \sin \vartheta \sin\left(\sum_{m=1}^{\infty} v_m \sin m\vartheta\right)$$

Expanding in Maclaurin's series and again carrying out the multiplications of the Fourier series gives

$$\cos v = \sum_{m=0}^{\infty} C_m \cos m \frac{\varphi - \varphi_p}{\varphi_s} \qquad \text{(E26)}$$

where, to the fourth order in J,

$$\left.\begin{aligned}
C_0 &= -\frac{1}{2}v_1 - \frac{1}{4}v_1v_2 + \frac{1}{16}v_1^3 \ldots \\
C_1 &= 1 - \frac{1}{8}v_1^2 - \frac{1}{2}v_2 - \frac{1}{4}v_2^2 - \frac{1}{4}v_1v_3 + \frac{1}{192}v_1^4 + \frac{1}{8}v_1^2v_2 \ldots \\
C_2 &= \frac{1}{2}v_1 - \frac{1}{2}v_3 - \frac{1}{12}v_1^3 \ldots \\
C_3 &= \frac{1}{2}v_2 + \frac{1}{8}v_1^2 + \frac{1}{8}v_2^2 - \frac{1}{2}v_4 - \frac{1}{128}v_1^4 - \frac{3}{16}v_1^2v_2 \ldots \\
C_4 &= \frac{1}{2}v_3 + \frac{1}{4}v_1v_2 + \frac{1}{48}v_1^3 \ldots \\
C_5 &= \frac{1}{2}v_4 + \frac{1}{8}v_2^2 + \frac{1}{4}v_1v_3 + \frac{1}{16}v_1^2v_2 + \frac{1}{384}v_1^4 \ldots
\end{aligned}\right\} \qquad \text{(E27)}$$

The final series is the generalization of Kepler's equation (eq. (72)). The problem is to evaluate the integrals

$$\beta_j \equiv \int_0^v \frac{\cos^{2j}(\psi/2)}{(1+\epsilon\cos\psi)^2} d\psi$$

$$\delta_j \equiv \int_0^v \frac{\sin^{2j}(\psi/2)}{(1+\epsilon\cos\psi)^2} d\psi$$

To do this introduce the additional pair of integrals

$$\alpha_j \equiv \int_0^v \frac{\cos^{2j}(\psi/2)}{1+\epsilon\cos\psi} d\psi$$

$$\gamma_j \equiv \int_0^v \frac{\sin^{2j}(\psi/2)}{1+\epsilon\cos\psi} d\psi$$

By direct integration,

$$\alpha_0 = \gamma_0 = \frac{E}{\sqrt{1-\epsilon^2}}$$

$$\beta_0 = \delta_0 = \frac{E - \epsilon \sin E}{(1-\epsilon^2)^{3/2}}$$

Now consider

$$\begin{aligned}
\alpha_{j+1} &= \int_0^v \frac{\cos^{2j}(\psi/2)\cos^2(\psi/2)}{1+\epsilon\cos\psi} d\psi \\
&= \frac{1}{2}\int_0^v \frac{\cos^{2j}(\psi/2)(1+\cos\psi)}{1+\epsilon\cos\psi} d\psi \\
&= \frac{1}{2}\int_0^v \frac{\cos^{2j}(\psi/2)\left(\frac{1}{\epsilon}+\cos\psi+1-\frac{1}{\epsilon}\right)}{1+\epsilon\cos\psi} d\psi \\
&= \frac{1}{2\epsilon}\int_0^v \cos^{2j}(\psi/2)d\psi + \frac{\epsilon-1}{2\epsilon}\alpha_j
\end{aligned}$$

The same method gives the analogous recurrence relations

$$\beta_{j+1} = \frac{1}{2\epsilon}\alpha_j + \frac{\epsilon-1}{2\epsilon}\beta_j$$

$$\gamma_{j+1} = -\frac{1}{2\epsilon}\int_0^v \sin^{2j}(\psi/2)d\psi + \frac{\epsilon+1}{2\epsilon}\gamma_j$$

$$\delta_{j+1} = -\frac{1}{2\epsilon}\gamma_j + \frac{\epsilon+1}{2\epsilon}\delta_j$$

Thus, the successive coefficients, M_s, in Kepler's equation can be computed recursively. The first few are (upper sign to be used if $\sigma>0$, lower sign if $\sigma<0$):

$$\left.\begin{aligned}
M_0&=E-\epsilon\sin E\\
M_1&=\frac{1\mp\epsilon}{4}\,(E\pm\sin E)\\
M_2&=3\,\frac{(1-\epsilon^2)^{3/2}v-(1\mp\epsilon)^2[(1\pm2\epsilon)E+\epsilon\sin E]}{32\epsilon^2}\\
M_3&=\frac{5}{128\epsilon^3}\,\{(1-\epsilon^2)^{3/2}[(3\epsilon\mp2)v\pm\epsilon\sin v]+(1\mp\epsilon)^3[(3\epsilon\pm2)E\pm\epsilon\sin E]\}\\
M_4&=\frac{35}{8192\epsilon^4}\,\{(1-\epsilon^2)^{3/2}[(26\epsilon^2\mp32E+12v)+8\epsilon(\pm2\epsilon-1)\sin v+\epsilon^2\sin 2v]-4(1\mp\epsilon)^4[(3\pm4\epsilon)E+\epsilon\sin E]\}
\end{aligned}\right\}\quad\text{(E28)}$$

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